Moldovan Team Selection Tests 2006

First Test

1. Determine all even numbers $n \in \mathbb{N}$ such that
$$\frac{1}{d_1}+\frac{1}{d_2}+\cdots+\frac{1}{d_k}=\frac{1620}{1003},$$
where $\{d_1,\dots,d_k\}$ is the set of divisors of n.

2. Consider a right-angled triangle ABC with the hypothenuse AB of size 1. The bisector $\angle ACB$ intersects the medians BE and AF at P and M, respectively. If $AF \cap BE = P$, determine the maximal value for S_{MNP}.

3. Let a, b, c be the sides of a triangle. Prove that
$$a^2\left(\frac{b}{c}-1\right)+b^2\left(\frac{c}{a}-1\right)+c^2\left(\frac{a}{b}-1\right)\geq 0.$$

4. Assume that m circles pass through points A and B. We start by labeling the points A and B by 1. In the second step we label every midpoint of an open arc AB with 2. Every subsequent step is performed as follows: For any two points that are already labeled by a and b, and are consecutive on some of the arcs we label the midpoint of that arc by $a+b$. We repeat the procedure n times and denote by $r(n,m)$ the number of appearances of the number n.

   (a) Determine $r(n,m)$.

   (b) For $n=2006$, find the smallest m for which $r(n,m)$ is a perfect square.

   Example of steps for the half-arc:
   $1-1$;
   $1-2-1$;
   $1-3-2-3-1$;
   $1-4-3-5-2-5-3-4-1$;
   $\vdots$

Second Test

1. Let (a_n) be the Lucas sequence defined as: $a_0=2$, $a_1=1$, $a_{n+1}=a_n+a_{n-1}$ for $n\geq 1$. Show that a_{59} divides $(a_{30})^{59}-1$.

2. Let C_1 be a circle in the interior of the circle C_2. Let P be a point in the interior of C_1 and Q a point in the exterior of C_2. Variable lines l_i are drawn through P in such a way to not contain Q. Assume that l_i intersect C_1 in A_i and B_i. Assume that the circumcircle of QA_iB_i intersect C_2 in M_i and N_i. Prove that all the lines M_iN_i are concurrent.

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3. Let a, b, c be sides of a triangle and p its semiperimeter. Prove that

$$a\sqrt{\frac{(p-b)(p-c)}{bc}}+b\sqrt{\frac{(p-c)(p-a)}{ca}}+c\sqrt{\frac{(p-a)(p-b)}{ab}}\geq p.$$

4. Let $A=\{1,2,\dots,n\}$. Find the number of unordered triples (X,Y,Z) such that $X\cup Y\cup Z=A$.

Third Test

1. Given a point P in the interior of $\triangle ABC$, assume that the rays AP, BP, and CP intersect the circumcircle of $\triangle ABC$ at A_1, B_1, and C_1. Prove that the maximal value for the sum of the areas A_1BC, B_1AC, and C_1AB is $p(R-r)$, where p, r, and R are the semi-perimeter, inradius, and circumradius of $\triangle ABC$, respectively.

2. Let $n\geq 2$ be an integers and X a set with $n+1$ elements. The ordered sequences $(a_1,a_2,\dots,a_n)$ and $(b_1,\dots,b_n)$ of distinct elements of X are said to be *separated* if there exist indeces $i\neq j$ such that $a_i=b_j$. Determine the maximal number of ordered sequences of n elements of X such that any two of them are separated.

3. Positive real numbers a, b, c satisfy the relation $abc=1$. Prove that

$$\frac{(a+3)}{(a+1)^2}+\frac{(b+3)}{(b+1)^2}+\frac{(c+3)}{(c+1)^2}\geq 3.$$

4. Denote by $f(n)$ the number of permutations $(a_1,\dots,a_n)$ of the set $\{1,2,\dots,n\}$ which satisfy the conditions $a_1=1$, and $|a_i-a_{i+1}|\leq 2$ for any $i=1,2,\dots,n-1$. Prove that $f(2006)$ is divisible by 3.

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