Chinese IMO Team Selection Test 2009

Time: 4.5 hours each day.

First Day

1. Let ABC be a triangle. Point D lies on the side BC such that $\angle CAD = \angle CBA$. Circle k passing through B and D intersects AB, AD at E and F respectively. Assume that BF intersects DE at G. Denote by M the midpoint of AG. Show that $CM \perp AO$.

2. Given an integer $n \geq 2$. Find the maximal constant $\lambda(n)$ such that: If a sequence $a_0, a_1, \dots, a_n$ of real numbers satisfies $0 = a_0 \leq a_1 \leq \cdots \leq a_n$, and $a_i \geq \frac{1}{2}(a_{i+1} + a_{i-1})$, $i = 1, 2, \dots, n-1$, then
$$\left(\sum_{i=1}^{n} i a_i\right)^2 \geq \lambda(n) \sum_{i=1}^{n} a_i^2.$$

3. Prove that for any odd prime p the number of positive integers n satisfying $p \mid n! + 1$ is smaller than or equal to $cp^{2/3}$ where c is a constant independent of p.

Second Day

4. a and b are positive real numbers such that $b - a > 2$. Prove that for any two distinct integers $m, n \in [a, b)$ there exists non-empty set $S \subseteq [ab, (a+1)(b+1))$ such that $\frac{1}{mn} \cdot \prod_{x \in S} x$ is a square of a rational number.

5. Let m, n be integers such that n is odd and satisfies $3 \leq n < 2m$. Numbers $a_{i,j}$ ($i, j \in \mathbb{N}$, $1 \leq i \leq m, 1 \leq j \leq n$) satisfy:

   1° For any $1 \leq j \leq n$, $a_{1,j}, a_{2,j}, \dots, a_{m,j}$ is a permutation of $1, 2, \dots, m$;

   2° For any $1 < i \leq m$, $q \leq j \leq n-1$, $|a_{i,j} - a_{i,j+1}| \leq 1$.

   Find the minimal value of $\max_{1<i<m} \sum_{j=1}^{n} a_{i,j}$.

6. Determine whether there exists an arithmetic progression consisting of 40 terms each of which can be written in the form $2^m + 3^n$ for some integers m, n.

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