# Bulgarian Mathematical Olympiad 1991, III Round

*First Day*

1. Prove that if $x_1, x_2, \ldots, x_k$ are mutally different (there are no two equal) (it is not required to be sequential) members of the arithmetic progression $2, 5, 8, 11, \ldots$ for which:
$$\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_k} = 1$$
then its number is greater than 8 ($k \geq 8$).

2. On the hypotenuse $AB$ of a right-angled triangle $ABC$ is fixed a point $P$. Let $\ell^{\rightarrow}$ is a ray formed by the line $BC$ with starting point $C$ and not containing the point $B$. For each point $T \neq C$ from $\ell^{\rightarrow}$ with $S$ is denoted the intersection point of the lines $PT$ and $AC$ and $M$ is the intersection point of the lines $BS$ and $AT$. Find the locus of the point $M$ when $T$ is moving on the ray $\ell^{\rightarrow}$.

3. Let $Oxy$ is a right-angled plane coordinate system. A point $A(x,y)$ is called rational if its coordinates are rational numbers (for example the point $A_0(-1,0)$ is a rational point). Let $k$ is a circle with the beginning of the coordinate system and with radii 1.

   (a) Prove that $A(x,y) \neq A_0$ is a rational point from $k$ if and only if
$$x = \frac{1-p^2}{1+p^2}, \qquad y = \frac{2p}{1+p^2}$$
   for some rational number $p$.

   (b) Find an infinite sequence $A_1(x_1,y_1), A_2(x_2,y_2), \ldots, A_n(x_n,y_n), \ldots$ that consists from mutually different points from $k$ in such a way that $\lim\limits_{n\to\infty} A_n = A_0$ (i. e. $\lim\limits_{n\to\infty} x_n = -1$ and $\lim\limits_{n\to\infty} y_n = 0$ and the length of the segment $A_nA_0$ is a rational number for: $n = 1, 2, \ldots$.

   (c) Prove that for each arc from $k$ we may choose at least two rational points such that the length of the distance between them is a rational number.

*Second day*

4. (a) Prove that if $a$, $b$, $c$ are positive real numbers for which the following inequality is satisfied
$$\left(a^2+b^2+c^2\right)^2 > 2\left(a^4+b^4+c^4\right)$$
   then there exists a triangle with sides $a$, $b$ and $c$.



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(b) Prove that if $a$, $b$, $c$, $d$ are positive real numbers for which the following inequality is satisfied

$$\left(a^2+b^2+c^2+d^2\right)^2 > 3\left(a^4+b^4+c^4+d^4\right)$$

then can be formed a triangle with sides equal to any three from the numbers given.

5. Prove that if from the angle bisectors of a given triangle can be constructed a triangle similar to the triangle given. Then the triangle given is an equilateral triangle.

6. Let the natural number $n \geq 3$ is presented as a sum of $k \geq 2$ natural numbers

$$n = x_1 + x_2 + \cdots + x_k$$

in such a way that $x_i \leq \frac{n}{2}$ for each $i = 1, 2, \ldots, k$. Prove that the vertices of an $n$-gon can be colored in $k$-colors in such a way that $x_1$ vertices are colored in the first color, $x_2$ vertices are colored in the second color, $\ldots, x_k$ vertices are colored in the $k$-th color and every two different vertices are colored in different colors.



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