24-th Iberoamerican Mathematical Olympiad

September 22–23, 2009

First Day

1. Given a positive integer $n \geq 2$, consider a set of n islands $X_1, \ldots, X_n$ arranged in a circle. Two bridges are built between each pair of neighboring islands.

   Starting from the island X_1, in how many ways one can cross the $2n$ bridges so that no bridge is crossed more than once?

2. Let n be a positive integer. Denote by a_n the largest of the positive integers m such that $2^{2^m} \leq n2^n$. Find all numbers that don't belong to the sequence $(a_n)_{n=1}^{\infty}$.

3. Let C_1 and C_2 be two congruent circles with centers O_1 and O_2, which intersect at A and B. Let P be a point of the arc AB of C_2 which is contained in the interior of C_1. AP intersects C_1 at C, CB intersects C_2 at D, and the bisector of $\angle CAD$ intersects C_1 and C_2 at E and L, respectively. Let F be the symmetric point to D with respect to the midpoint of PE. Prove that there exists a point X satisfying $\angle XFL = \angle XDC = 30^\circ$ and $CX = O_1O_2$.

Second Day

4. Given a triangle ABC with incenter I, let P be the intersection point of the external bisector of $\angle A$ and the circumcircle of $\triangle ABC$. Let J be the second intersection point of PI and the circumcircle of $\triangle ABC$. Show that the circumcircles of $\triangle JIB$ and $\triangle JIC$ are tangent to IC and IB, respectively.

5. Consider the sequence $\{a_n\}_{n=1}^{\infty}$ defined as follows: $a_1 = 1$, $a_{2k} = 1 + a_k$, and $a_{2k+1} = \frac{1}{a_{2k}}$ for $k \geq 1$. Prove that every positive rational number appears in the sequence $\{a_n\}$ exactly once.

6. 6000 points are marked on a circle, and they are colored using 10 colors in such a way that within every group of 100 consecutive points all the colors are used. Determine the least positive integer k with the following property: In every coloring satisfying the above condition, it is possible to find a group of consecutive points in which all the colors are used.

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