Chinese IMO Team Selection Test 2008

Time: 4.5 hours each day.

First Day

1. Let $ABC$ be a triangle such that $AB > AC$. Let $E$ be the point of tangency of $BC$ with the incircle of $ABC$. Let $D$ be the second intersection point of the incircle with the segment $AE$. Point $F \in AE$ ($F \neq E$) satisfies $CE = CF$. The ray $CF$ intersects $BD$ at $G$. Prove that $CF = FG$.

2. The sequence $(x_n)$ is defined by $x_1 = 2$, $x_2 = 12$, and $x_{n+2} = 6x_n + 1 - x_n$, for $n \geq 1$. Let $p$ be an odd prime number and let $q$ be a prime divisor of $x_p$. Prove that if $q \notin \{2, 3\}$ then $q \geq 2p - 1$.

3. Suppose that every positive integer has been painted in one of the two colors. Prove that there exists an infinite sequence of positive integers $a_1 < a_2 < \cdots$ such that $a_1$, $a_1 + a_2$, $a_2$, $a_2 + a_3$, $a_3$, … is an infinite sequence of positive integers of the same color.

Second Day

4. Let $n \geq 4$ be an integer. Consider all the subsets of $G_n = \{1, 2, \ldots, n\}$ with at least two elements. Prove it is possible to arrange those subsets in a sequence $P_1, P_2, \ldots, P_{2^n-n-1}$ such that $|P_i \cap P_{i+1}| = 2$ for every $i = 1, 2, \ldots, 2^n - n - 2$.

5. Let $m, n$ be two positive integers. Positive real numbers $a_{i,j}$ ($1 \leq i \leq n$, $1 \leq j \leq m$) are not all equal to zero. If

$$f = \frac{n \sum_{i=1}^{n} \left( \sum_{j=1}^{m} a_{ij} \right)^2 + m \sum_{j=1}^{m} \left( \sum_{i=1}^{n} a_{ij} \right)^2}{\left( \sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij} \right)^2 + mn \sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij}^2},$$

find the maximum and minimum of $f$.

6. Find the maximal constant $M$ such that for arbitrary integer $n \geq 3$, there exists two sequences of positive real numbers $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ such that

(i) $\sum_{k=1}^{n} b_k = 1$, $2b_k \geq b_{k-1} + b_{k+1}$, $k = 2, 3, \ldots, n - 1$;

(ii) $a_k^2 \leq 1 + \sum_{i=1}^{k} a_i b_i$, $k = 1, 2, \ldots, n$, $a_n = M$. 

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