Chinese IMO Team Selection Test 2007

First Day

1. Points $A$ and $B$ lie on a circle $k$ with center $O$. Let $C$ be a point outside the circle and let $CS$ and $CT$ be the tangents to the circle. Let $M$ be the midpoint of the smaller arc $AB$ of $k$. The lines $MS$ and $MT$ intersect $AB$ at $E$ and $F$ respectively. The lines passing through $E$ and $F$ perpendicular to $AB$ intersect $OS$ and $OT$ at $X$ and $Y$ respectively. A line passing through $C$ intersects the circle $k$ at $P, Q$ ($P \in CQ$). Let $R$ be the intersection of $MP$ with $AB$, and let $Z$ be the circumcenter of $\triangle PQR$. Prove that $X, Y,$ and $Z$ are collinear.

2. A natural number $x$ is called good if it satisfies: $x = p/q > 1$ with $p, q \in \mathbb{N}$, $(p, q) = 1$, and there exist constants $\alpha, N$ such that for any integer $n \geq N$,

$$|\{x^n\} - \alpha| \leq \frac{1}{2(p + q)}.$$

Find all good numbers.

3. There are 63 points on a circle with diameter 20. Let $S$ be the number of triangles whose vertices are three of the 63 points and side lengths are $\geq 9$. Find the maximum of $S$.

Second Day

4. Find all functions $f : \mathbb{Q}^+ \to \mathbb{Q}^+$ such that

$$f(x) + f(y) + 2xyf(xy) = \frac{f(xy)}{f(x+y)}.$$

5. Let $x_1, \ldots, x_n$ $(n > 1)$ be real numbers satisfying $A = |\sum_{i=1}^n x_i| \neq 0$ and $B = \max_{1 \leq i < j \leq n} |x_j - x_i| \neq 0$. Prove that for any $n$ vectors $\vec{\alpha}_i$ in the plane, there exists a permutation $(k_1, \ldots, k_n)$ of the numbers $1, 2, \ldots, n$ such that

$$\left| \sum_{i=1}^k x_{k_i} \vec{\alpha}_{k_i} \right| \geq \frac{AB}{2A + B} \max_{1 \leq i \leq n} |\vec{\alpha}_i|.$$

6. Let $n$ be a positive integer and let $A \subseteq \{1, 2, \ldots, n\}$. Assume that for any two numbers $x, y \in A$ the least common multiple of $x$ and $y$ is not greater than $n$. Show that $|A| \leq 1.9\sqrt{n} + 5$. 

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