Yugoslav IMO Team Selection Test 1980

Kumrovec, 1980

1. Circles k and l intersect at points P and Q. Let A be an arbitrary point on k distinct from P and Q. Lines AP and AQ meet l again at B and C. Prove that the altitude from A in triangle ABC passes through a point that does not depend on A.

2. Let a,b,c,m be integers, where $m > 1$. Prove that if

$$a^n + bn + c \equiv 0 \pmod{m}$$

for each natural number n, then $b^2 \equiv 0 \pmod{m}$. Must $b \equiv 0 \pmod{m}$ also hold?

3. A sequence (x_n) satisfies $x_{n+1} = \dfrac{x_n^2 + a}{x_{n-1}}$ for all $n \in \mathbb{N}$. Prove that if x_0, x_1 and $\dfrac{x_0^2 + x_1^2 + a}{x_0 x_1}$ are integers, then all the terms of sequence (x_n) are integers.

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