39-th Yugoslav Federal Mathematical Competition 1998

High School

Bečej, April 11, 1998

Time allowed 4 hours.
Each problem is worth 25 points.

1-st Grade

1. Prove that $\sqrt{2}(\sqrt[3]{3}+2)+\sqrt[3]{9} > 4\sqrt{2}\sqrt[3]{9}$.

2. Find mutually distinct decimal digits a,b,c,d such that $\overline{abccba} = \overline{cda}^2$.

3. The bisector of the angle A of an acute-angled triangle ABC meets side BC at D. If $AD = AB$ and $AD \perp OH$, where O is the circumcenter and H the orthocenter of the triangle, find the angles of triangle ABC.

4. Let $S = \{1,2,3,4,5,6\}$. How many bijective functions $f : S \to S$ are there such that for each $x \in S$, $f(f(x)) - x$ is divisible by 3?

2-nd Grade

1. Show that there is no positive integer n such that $8^n + 2^n + 1$ is a perfect square.

2. A circle k is tangent to the sides $A_1A_2, A_2A_3, \dots, A_nA_1$ of a convex $n-$gon $A_1A_2\dots A_n$ at points $B_1, B_2, \dots, B_n$, respectively. Let M be an arbitrary point on circle k. Prove that the product of the distances from M to lines $A_1A_2, A_2A_3, \dots, A_nA_1$ is equal to the product of the distances from M to lines $B_1B_2, B_2B_3, \dots, B_nB_1$.

3. Let $n > 1$ be a natural number and let

$$S_1 = \left[\sqrt{n}\right] + \left[\sqrt{2n}\right] + \left[\sqrt{3n}\right] + \cdots + \left[\sqrt{(n-1)n}\right] \quad \text{and}$$
$$S_2 = \left[\frac{1}{n}\right] + \left[\frac{4}{n}\right] + \left[\frac{9}{n}\right] + \cdots + \left[\frac{(n-1)^2}{n}\right].$$

   Prove that $S_1 + S_2 \geq (n-1)^2$, and show that equality holds if and only if n is not divisible by any perfect square greater than 1.

4. Let be given a sequence of points $A_i(x_i, y_i)$ with $0 < x_i, y_i < 1$ for $i = 1,2,\dots,n^2$. Show that there exists a permutation $(i_1, i_2, \dots, i_{n^2})$ of numbers $1,2,\dots,n^2$ such that the length of the polygonal line $A_{i_1}A_{i_2}\dots A_{i_{n^2}}$ does not exceed $2n+1$.

3-rd and 4-th Grades

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1. Let $a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n$ be real numbers. Prove that

$$\left(\sum_{i\neq j} a_i b_j\right)^2 \geq \left(\sum_{i\neq j} a_i a_j\right) \left(\sum_{i\neq j} b_i b_j\right).$$

2. Determine the smallest natural number n with the following property: There is a permutation of the set $\{0,1,2,\dots,9\}$ such that the sum of any three consecutive terms of this permutation is at most n.

3. More than half of faces of a convex polyhedron can be colored so that no two colored faces share an edge. Prove that a sphere cannot be inscribed in this polyhedron.

4. Let n be a natural number greater than 4. Prove that the following two conditions are equivalent:

   (i) both n and $n+1$ are composite numbers;

   (ii) the closest integer to $\dfrac{(n-1)!}{n^2+n}$ is even.

   (The closest integer to $k+1/2$, where $k \in \mathbb{Z}$, is defined to be $k+1$.)