37-th Yugoslav Federal Mathematical Competition 1996

High School

Bar, April 13, 1996

Time allowed 4 *hours.*
Each problem is worth 25 *points.*

1-st Grade

1. A city of the square form has 22 streets which lie on the lines $x = n$ and $y = k$, $n,k \in \{0,1,\dots,10\}$. Five businessmen having offices at points $A(0,43/5)$, $B(17/5,2)$, $C(16/5,9)$, $D(17/2,0)$, $E(46/5,8)$ want to make a common meeting at some of the city crossings. Which crossing should be chosen or the meeting place if they want the sum of distances to the offices to be minimal? (Note: the businessmen walk only along streets.)

2. Let P and Q be points on sides BC and CD of a square $ABCD$ such that PQ is a tangent of the circle with the center A and radius AB. Segments AP and AQ intersect the diagonal BD at R and S. Prove that points C,P,Q,R and S lie on a circle.

3. Let $ABCD$ be a square, E be the point on ray AD such that $DE = AD$ and l be the smaller arc AE of the circle with the center C and radius CA. Let $\mathscr{F}$ be the figure bounded by segments AB, BC, CD, DE and arc l. Divide $\mathscr{F}$ by a polygonal line into two connected congruent parts.

4. Find seven distinct primes $p_1, p_2, \dots, p_7$ less than 1000 satisfying
$$p_2 - p_1 = p_3 - p_2 = \cdots = p_7 - p_6.$$

2-nd Grade

1. Prove that there exists a system of 1996 circles in the plane such that any two circles have at most two points in common and each circle touches exactly five others.

2. Find the locus of barycenters of all equilateral triangles inscribed in a given square.

3. Find all solutions in nonnegative integers to the equation
$$\sqrt{x} + \sqrt{y} + \sqrt{z} = \sqrt{1996}.$$

4. Determine the maximal real number a for which the inequality
$$x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 \geq a(x_1x_2 + x_2x_3 + x_3x_4 + x_4x_5)$$
holds for any five real numbers x_1, x_2, x_3, x_4, x_5.

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3-rd and 4-th Grades

1. Given nonnegative integers a,b,c, prove that there is no function $f:\mathbb{N}\to\mathbb{N}$ such that
$$f(x+y)+f(x)+f(y)=xy+ax+by+c$$
holds for each $x,y\in\mathbb{N}$.

2. Let O be the intersection of diagonals of a convex quadrilateral $ABCD$ and let P and Q be the circumcenters of triangles ABO and CDO respectively. Prove the inequality
$$AB+CD\leq 4PQ.$$

3. Given $n\in\mathbb{N}$, determine the maximal positive integer k for which there exists a k-element subset A of $\{1,3,\dots,2n-1\}$ which does not contain any two numbers one of which divides the other.

4. Among 1996 almost identical balls two are defective. All regular balls have equal weights. Weights of the defective balls are equal, but different from the weight of a regular ball. Prove that one can find out whether defective balls are lighter or heavier than regular balls by at most three measurements on a scale without weights.

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