36-th Yugoslav Federal Mathematical Competition 1995

High School

Vrbas, April 15, 1995

Time allowed 4 hours.
Each problem is worth 25 points.

1-st Grade

1. How many five-element subsets S of set $A = \{0,1,2,\dots,9\}$ are there which satisfy
$$\{r(x+y) \mid x,y \in S,\ x \neq y\} = A,$$
where $r(n)$ denotes the remainder when n is divided by 10?

2. Let ABC be an acute-angled triangle. Let D be the foot of the altitude from C, E be the foot of the perpendicular from D to AC, and F be the point on segment DE such that $DF : FE = DA : DB$. Prove that lines BE and CF are mutually perpendicular.

3. A regular $1995-$gon is inscribed in a circle. From a point P on the circle one draws chords joining it to every segment of the $1995-$gon. Prove that the sum of some 1000 of these chords is equal to the sum of the remaining 995 chords.

4. Show that there exists a set S of 1995 distinct natural numbers with the following two properties:
   (i) The sum of two or more distinct numbers from S is always a composite number.
   (ii) The numbers in S are pairwise coprime.

2-nd Grade

1. Show that the number $2^{2^{1995}} - 1$ has at least 1995 distinct prime factors.

2. A convex hexagon $ABCDEF$ is inscribed in a circle. Prove that if $AB \cdot CD \cdot EF = BC \cdot DE \cdot FA$, then the diagonals AD, BE and CF meet in a point.

3. Let $\mathscr{M}$ be a convex polygon of perimeter p. Show that the set of sides of $\mathscr{M}$ can be partitioned into two disjoint subsets A and B such that
$$|s_A - s_B| \leq \frac{p}{3},$$
where s_A, s_B respectively denote the sums of the lengths of the sides in A and B.

4. A square 5×5 is divided into 25 unit squares. Players A and B alternately write numbers in the unit squares. Player A begins and always writes 1, and player B always writes 0. When 25 numbers are written, one computes the sums of numbers in all squares 3×3 and denotes by M the largest of these sums.

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(a) Player A can always achieve that $M \geq 6$.

(b) Player B can always achieve that $M \leq 6$.

3-rd and 4-th Grades

1. If p is a prime number, prove that the number

$$\underbrace{11\dots1}_{p}\underbrace{22\dots2}_{p}\dots\underbrace{99\dots9}_{p}-123456789$$

is divisible by p.

2. We say that a polynomial $P(x)$ with integer coefficients is divisible by a natural number m if $P(a)$ is divisible by m for every integer a. Prove that if a polynomial $P(x) = a_0x^n + \cdots + a_{n-1}x + a_n$ is divisible by m, then $n!a_0$ is also divisible by m.

3. A chord AB and a diameter CD of a circle k are mutually perpendicular and intersect at M. Let P be a point on the arc ACB, distinct from A, B, C. Line PM meets k again at point Q, and line PD meets AB at R. Prove that $RD > MQ$.

4. Let P and Q be the midpoints of edges AB and CD of a tetrahedron $ABCD$ and let O and S be the circumcenter and incenter of the tetrahedron, respectively. Prove that if points P, Q, S lie on a line, then point O also belongs to that line.