# 33-nd Yugoslav Federal Mathematical Competition 1992

## High School

Podgorica, April 18, 1992

*Time allowed* 4 *hours.*
*Each problem is worth* 25 *points.*

### 1-st Grade

1. Given a triangle $ABC$ such that $\angle A = 60^\circ$, let $H$ be its orthocenter, $P$ and $Q$ be respectively midpoints of $BH$ and $CH$, $M$ the intersection of perpendicular on $BH$ from $P$ with $AB$ and $N$ the intersection of perpendicular on $CH$ from $Q$ with $AC$.
   (a) Prove that $M$, $H$ and $N$ are collinear.
   (b) Prove that $MN$ contains the center $O$ of the circumscribed circle of $ABC$.

2. Solve the system of equations in the set of real numbers:
$$\frac{2x_1^2}{1+x_1^2} = x_2,\ \frac{2x_2^2}{1+x_2^2} = x_3,\ \frac{2x_3^2}{1+x_3^2} = x_1.$$

3. $n$ points are given on a line. Determine the point $X$ on that line such that the sum of the distances from $X$ to the given points is the least possible.

4. On the $8\times 8$ chessboard 21 rectangles $3\times 1$ are placed such that only one unit square is uncovered. Find all possibilities for uncovered square.

### 2-nd Grade

1. Given a parallelogram $ABCD$ and a circle which contains $A$ and intersects $AB$, $BC$, $AD$, respectively at $B_1$, $C_1$, $D_1$, prove that
$$AB\cdot AB_1 + AD\cdot AD_1 = AC\cdot AC_1.$$

2. Let $ABC$ be isosceles triangle such that $AB = BC$ and $\angle B = 20^\circ$. Let $P$ and $Q$ be the points on the sides $BC$ and $AB$ respectively such that $\angle PAC = 50^\circ$ and $\angle QCA = 60^\circ$. Determine $\angle PQC$.

3. In the chess tournament only masters and grandmasters have participated. Each two contestants have played exactly one game. It is known that each contestant has won exactly half of his points in the games with masters. Prove that the number of participants is the exact square of a natural number.



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4. Two players play the following game: first player writes down one digit; after his move second player write one digit to the left or write to previously written digit; after his move first player write one digit to the left or write to previously written number, etc. Prove that the first player can play in that way such that after each of the moves of the second player, obtained number is not the exact square.

**3-rd and 4-th Grades**

1. Determine all natural numbers $n$ such that the following equality holds:

$$\cos\frac{\pi}{7}-\cos\frac{2\pi}{7}+\cos\frac{3\pi}{7}-\cdots+(-1)^{n-1}\cos\frac{n\pi}{7}=\frac{1}{2}.$$

2. Given the natural numbers $a_1, a_2, \ldots, a_n$ such that:

   (a) $a_1 = 1$;

   (b) $a_i \le a_{i+1} \le 2a_i$ for $i = 1, 2, \ldots, n-1$;

   (c) $a_1 + a_2 + \cdots + a_n = 1992$.

   Is it possilbe to partition these numbers into two sets, such that the sum of the numbers in the first set is equal to the sum of the numbers in the second set?

3. Prove that for each integer $n \ge 2$ there are $n$ mutually distinct natural numbers such that the sum of each two of them is divisible by their difference.

4. Let $R$ be the radius of circumscribed, $r$ the radius of inscribed circle and $h$ the longest altitude of an triangle. Prove that $R + r \le h$.



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