32-nd Yugoslav Federal Mathematical Competition 1991

High School

Valjevo, April 20, 1991

1-st Grade

1. A line passing through the centroid T of a triangle ABC meets sides AB and AC and line BC at points P,Q,R respectively, where C lies on the segment BR. Prove that
$$\frac{1}{TP}=\frac{1}{TQ}+\frac{1}{TR}.$$

2. A figure $\mathscr{F}$ consists of points B and D and the two arcs BD constructed in the interior of the square $ABCD$ with centers A and C and radii equal to a side of the square. Show that all rectangles circumscribed about the figure $\mathscr{F}$ (with each side having exactly one point in common with $\mathscr{F}$) have equal perimeter.

3. Does there exist a sequence of length 3982 with the following properties:

   (i) Each $k\in\{1,2,\dots,1991\}$ appears in the sequence twice;

   (ii) For each k, there are exactly $k-1$ terms between the two appearances of k in the sequence?

4. If $a\geq 1$ and $b\geq 1$, prove the inequality:
$$3\left(\frac{a^2-b^2}{8}\right)^2+\frac{ab}{a+b}\geq\sqrt{\frac{a^2+b^2}{8}}.$$

2-nd Grade

1. A triangle and a square are circumscribed about a circle of diameter 1. Show that the part of the boundary of the square which lies outside the triangle has length smaller than 1.8.

2. Four lines in the plane, every two of which intersect and no three of which have a common point, determine four triangles.

   (a) Prove that the circumcircles of these four triangles have a common point X.

   (b) Prove that the circumcenters of the four triangles lie on a circle that contains point X.

3. If a,b,c are positive numbers, prove the inequality
$$\frac{1}{a^3+b^3+abc}+\frac{1}{b^3+c^3+abc}+\frac{1}{c^3+a^3+abc}\leq\frac{1}{abc}.$$
When does equality hold?

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4. In the plane are given $2n$ points, no three of which are collinear; n of these points are colored blue and n are colored red. Show that there exist n segments, each of which joins one blue and one red point, such that no two of these segments intersect.

3-rd and 4-th Grades

1. Let a be a natural number and let (x_n) be the sequence defined as follows: $x_1 = a$ and, for each $n \in \mathbb{N}$,

$$x_{n+1} = \begin{cases} \dfrac{x_n}{2}, & \text{if } x_n \text{ is even;} \\ \dfrac{3x_n+1}{2}, & \text{if } x_n \text{ is odd.} \end{cases}$$

Show that at least one member of the sequence (x_n) is even.

2. Find the number of permutations $(a_1, a_2, \ldots, a_n)$ of $1, 2, \ldots, n$ such that, for every $k = 2, 3, \ldots, n$, at least one of the numbers $a_1, a_2, \ldots, a_{k-1}$ differs from a_k by 1.

3. Suppose α is a zero of the polynomial $p(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n$, where $a_0 \geq a_1 \geq \cdots \geq a_n > 0$. Prove that $|\alpha| \leq 1$.

4. Let be given a rectangular table with m columns and n rows, where $m > n$ and m, n are of the same parity. There is a white bishop in the lower-left corner and a black bishop in the upper-right corner. Players B and C alternately move the bishops in accordance with the chess rules. B plays first and always moves the white bishop, and C always moves the black bishop. The winner is a player who places his bishop under attack of the other bishop. Decide which player has a winning strategy and describe one such strategy.

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