31-st Yugoslav Federal Mathematical Competition 1990

High School

Tuzla, April 21, 1990

1-st Grade

1. For a natural number n, prove the identity:

$$\overline{\underbrace{33\ldots3}_{n}}^2+\overline{\underbrace{55\ldots5}_{n-1}\underbrace{44\ldots4}_{n}}^2=\overline{\underbrace{55\ldots5}_{n-1}\underbrace{44\ldots4}_{n-1}5}^2.$$

2. Let ABC be an acute-angled triangle with $\angle ACB = 60^\circ$. Prove that the circumcenter of the triangle lies on the bisector of one of the angles formed by the altitudes from A and B.

3. Let $a_1 < a_2 < \cdots < a_n$ be real numbers. Prove that if f is a bijection mapping $\{a_1,\ldots,a_n\}$ onto itself such that

$$a_1+f(a_1)<a_2+f(a_2)<\cdots<a_n+f(a_n),$$

then f is the identity.

4. Initially the numbers 1 and 2 are written on the board. We are permitted to write new numbers on the board according to the following rule: if numbers a and b are written on the board, then we can write the number $ab+a+b$. Can we manage to write number (a) 13121; (b) 12131?

2-nd Grade

1. Find all pairs (x,y) of natural numbers with the following property: exchanging the last two decimal digits of x^2 one obtains y^2.

2. A broken line whose every segment is of length 3 and whose all vertices lie on the surface of a cube of side 2, joins two opposite vertices of the cube. At least how many segments can such a broken line consist of?

3. Let ABC be a triangle. Points K and L are chosen on segments AB and BC respectively, and point M is chosen on segment KL. Prove that

$$\sqrt[3]{S_{ABC}} \geq \sqrt[3]{S_{AKM}} + \sqrt[3]{S_{MLC}},$$

where S_X denotes the area of X.

4. An imperor wants to build a castle with 1990 rooms in one floor so that:

   (i) In any room there are 0, 1 or 2 doors.

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(ii) Between any two rooms there is at most one door, and in any room there is at most one door to outside the castle.

(iii) The number of doors to outside is 19 and the number of rooms with exactly one door is 90.

Is it possible to build such a castle?

3-rd and 4-th Grades

1. A rectangular box $m \times n \times p$, where m, n, p are natural numbers, consists of mnp unit cubes. Each of these cubes is assigned a real number. For every rectangular box $m \times 1 \times 1$, $1 \times n \times 1$ or $1 \times 1 \times p$ consisting of these cubes, the numbers assigned to these boxes form an arithmetic sequence in that order. The sum of the eight numbers at the cubes at the vertices of the box is a. Compute the sum of all the assigned numbers.

2. Set $x_0 = 1990$ and
$$x_n = -\frac{1990}{n}(x_0 + x_1 + \cdots + x_{n-1}) \qquad \text{for } n \geq 1.$$
Calculate $x_0 + 2x_1 + 2^2 x_2 + \cdots + 2^{1990} x_{1990}$.

3. Let S and O be the incenter and orthocenter of a triangle ABC, respectively. Let Q be a point such that S is the midpoint of segment OQ and let T_1, T_2 and T_3 be the centroids of triangles BCQ, CAQ and ABQ, respectively. Prove that $AT_1 = BT_2 = CT_3 = \frac{4}{3}R$, where R is the circumradius of triangle ABC.

4. Find the number of relations on an n-element set which are neither symmetric nor antisymmetric? (A relation $\sim$ is said to be antisymmetric if $b \not\sim a$ whenever $a \sim b$.)