30-th Yugoslav Federal Mathematical Competition 1989

High School
Skopje, April 1989

1-st Grade

1. If x, y, z are positive real numbers with $x+y+z=1$, prove that
$$\left(1+\frac{1}{x}\right)\left(1+\frac{1}{y}\right)\left(1+\frac{1}{z}\right) \geq 64.$$
When does equality occur?

2. Suppose $x_1, x_2, \dots, x_{1990}$ are natural numbers such that
$$x_1^2+x_2^2+\cdots+x_{1989}^2=x_{1990}^2.$$
Prove that at least two of these numbers are even.

3. Construct triangle ABC, given that $BC=a$, $CA=b$ and $\angle CAB=3\angle ABC$.

4. Find all natural numbers n for which
$$\left[\sqrt[3]{1}\right]+\left[\sqrt[3]{2}\right]+\cdots+\left[\sqrt[3]{n}\right]=2n.$$

2-nd Grade

1. Let C, D be points on the semicircle with diameter AB, with C on the arc AD and $\angle CSD=90^\circ$, where S is the midpoint of AB. Let AC and BD meet at E and let AD and BC meet at F. Show that the vector $\overrightarrow{EF}$ does not depend on choice of C and D.

2. If $a \geq 1$, $b \geq 1$, $c \geq 1$, prove that
$$\sqrt{a-1}+\sqrt{b-1}+\sqrt{c-1} \leq \sqrt{c(ab+1)}.$$
When does equality hold?

3. Determine all triples (x, y, z) of integers such that $x^y-2^z=1$.

4. Let m and n be coprime positive integers. In each cell of an infinite chessboard a real number is written so that the sum of numbers in every $m \times n$ or $n \times m$ rectangle is equal to zero. Show that at least two of the written numbers are equal.

3-rd and 4-th Grades

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1. Find all quadruples (x_1,x_2,x_3,x_4) of positive numbers which satisfy

$$\begin{aligned} x_1+x_2^2+x_3^3 &= 3,\\ x_2+x_3^2+x_4^3 &= 3,\\ x_3+x_4^2+x_1^3 &= 3,\\ x_4+x_1^2+x_2^3 &= 3. \end{aligned}$$

2. Let $P(x)$ be a polynomial with real coefficients such that $P(x)\geq 0$ for each real x. Show that there are polynomials $Q(x)$ and $R(x)$ such that $P(x)=Q(x)^2+R(x)^2$ for all x.

3. Find the number of ordered triples of sets (A,B,C) such that

   (i) $A\cup B\cup C=\{1,2,\dots,n\}$,

   (ii) $A\cap B\cap C=\emptyset$,

   (iii) $A\cap B=\emptyset$.

4. Let $ABCDA_1B_1C_1D_1$ be a cube. Does there exist a line that intersects each of the lines AB, CC_1, A_1D_1, DB_1?