29-th Yugoslav Federal Mathematical Competition 1988

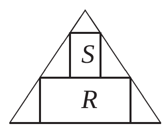
High School
Sinj, April 1988

1-st Grade

1. Prove that if n is a natural number which satisfies
$$\left[\frac{n}{1}\right]+\left[\frac{n}{2}\right]+\cdots+\left[\frac{n}{n}\right]=2+\left[\frac{n-1}{1}\right]+\left[\frac{n-1}{2}\right]+\cdots+\left[\frac{n-1}{n-1}\right],$$
then n is prime.

2. Find the angles of a triangle ABC in which the median, angle bisector and altitude from C divide $\angle ACB$ into four equal parts.

3. Two rectangles R and S are inscribed in an acute-angled triangle T as shown in the picture. Determine the maximum value of $\frac{P_R+P_S}{P_T}$, where P_X denotes the area of a figure X.



4. At an international conference two participants from each of 27 countries take part. Prove that the participants cannot be arranged around a round table so that between any two participants from the same country sit exactly 9 others.

2-nd Grade

1. Let O be the circumcenter of a triangle ABC. Let P,Q,R be the midpoints of arcs AB,BC,CA of the circumcircle not containing C,A,B respectively. If X is a point given by $\overrightarrow{OX}=\overrightarrow{OP}+\overrightarrow{OQ}+\overrightarrow{OR}$, show that X is the incenter of triangle ABC.

2. Determine the odd natural numbers $n\geq 3$ for which the function $f:\mathbb{Q}\to\mathbb{Q}$, $f(x)=x^n-2x$ is injective.

3. We say that a set $A\subset\mathbb{N}$ is *good* if for some natural number n the equation $x-y=n$ has infinitely many solutions (x,y) in A. Prove that if $\mathbb{N}=A_1\cup A_2\cup\cdots\cup A_{1988}$, then at least one of the sets $A_1,\dots,A_{1988}$ is good.

4. Prove that there are no two different points inside a convex $2n-$gon, each of which lies on n diagonals of the $2n-$gon.

3-rd and 4-th Grades

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1. Let $a,b,c,d \in \mathbb{N}_0$, $d \neq 0$. Prove that the function $f : \mathbb{N}_0 \to \mathbb{N}_0$, $f(x) = \left[\frac{ax+b}{cx+d}\right]$ is injective if and only if $c = 0$ and $a \geq d$.

2. Let be given an n-gonal pyramid in which a sphere can be inscribed. Each of the lateral faces is rotated around the corresponding base edge so as to have common interior points with the base. In this way, n different images of the vertex of the pyramid are obtained. Show that these n images lie on a circle.

3. Let $a_1, a_2, a_3, \dots$ be a strictly increasing sequence of natural numbers such that $a_1 = 1$, $a_2 = 2$ and $a_m a_n = a_{mn}$ for all coprime m and n. Prove that $a_n = n$ for all n.

4. There are more than seven cities in a country. Prove that there is no net of one-way roads with the following properties:

   (i) For any two cities there is exactly one road joining them.

   (ii) For any two cities A and B there is exactly one city which can be directly reached from both A and B.

   (iii) For any two cities A and B there is exactly one city from which one can directly reach both A and B.