27-th Yugoslav Federal Mathematical Competition 1986

High School

Postojna, April 1986

1-st Grade

1. Show that there exist infinitely many triples of consecutive natural numbers, each of which is a sum of two perfect squares. (Example: $72 = 6^2 + 6^2$, $73 = 8^2 + 3^2$, $74 = 7^2 + 5^2$.)

2. In a convex quadrilateral $ABCD$ it holds that $AB + BD \leq AC + CD$. Prove that $AB \leq AC$.

3. In triangle ABC, $\angle B = \angle C = 40^\circ$. Let D be the point on ray AB such that $AD = BC$. Compute the angles of triangle ADC.

4. On each cell of a 5×5 chessboard a marker is placed. A move consists of moving any two markers, each of them to a neighboring cell (two cells are neighboring if they shere a side). Consider some cell. Is it possible to place all the 25 markers in the considered cell after finitely many moves?

2-nd Grade

1. Suppose x and y are natural numbers satisfying $2x^2 + x = 3y^2 + y$. Prove that the numbers $x - y$, $2x + 2y + 1$ and $3x + 3y + 1$ are perfect squares.

2. Prove that for all positive numbers a, b, c,
$$\frac{a^3}{a^2+ab+b^2} + \frac{b^3}{b^2+bc+c^2} + \frac{c^3}{c^2+ca+a^2} \geq \frac{a+b+c}{3}.$$

3. Let C be a point on a diameter AA_1 of a circle. Let B be a point on the circle such that $AB = CA_1$. Prove that in the triangle ABC the angle bisector at A, the median at B and the altitude at C meet in a point.

4. Prove that among any five distinct positive numbers there are two such that neither their sum nor their absolute difference are not equal to any of the remaining three numbers.

3-rd and 4-th Grades

1. Find all functions $f : \mathbb{R} \to \mathbb{R}$ with the following properties:

   (a) $f(x + f(y)) = f(x + y) + 1$ for all real x, y;

   (b) f is strictly increasing.

The IMO Compendium Group,
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2. Prove that if $x_1, x_2, \dots, x_n$ are real numbers such that

$$(n-1)(x_1^2+x_2^2+\cdots+x_n^2)=(x_1+x_2+\cdots+x_n)^2,$$

then they are either all nonnegative or all nonpositive.

3. From the midpoint of each side of a cyclic quadrilateral, the perpendicular to the opposite side is drawn. Show that these four perpendiculars have a common point.

4. Find the greatest integer k with the following property: For any arrangement of the numbers $1,2,\dots,64$ in the cells of a 8×8 chessboard, there exist two neighboring cells such that the difference of numbers in these cells is not smaller than k. (Two cells are considered neighboring if they have a common vertex.)