26-th Yugoslav Federal Mathematical Competition 1985

High School

Cetinje, April 1985

1-st Grade

1. Prove that among any 39 consecutive natural numbers there is one whose sum of (decimal) digits is divisible by 11.

2. A point E is given in the interior of a square $ABCD$ such that triangle CDE is isosceles with $\angle E = 150^\circ$. Find the angles of triangle ABE.

3. Let 3000 points be given in the plane, no three of which lie on a line. Show that there exist 1000 triangles with vertices in these points such that no two of these triangles have a common point.

4. If a,b,c,d,e,f are positive real numbers, prove the inequality

$$\frac{ab}{a+b}+\frac{cd}{c+d}+\frac{ef}{e+f}\leq\frac{(a+c+e)(b+d+f)}{a+b+C+d+e+f}.$$

2-nd Grade

1. Determine the smallest natural number n for which the sums of digits of n and $n+1$ are both divisible by 1985.

2. The function $f:\mathbb{N}\to\mathbb{N}$ is given by the formula $f(m)=m+[\sqrt{m}]$. Prove that for each $m\in\mathbb{N}$ there exists $k\in\mathbb{N}$ such that $f^k(m)=\underbrace{f(f(\cdots f}_{k}(m)\cdots))$ is a perfect square.

3. Let Q be a point inside a tetrahedron $PABC$. Prove that

$$\angle BQC+\angle CQA+\angle AQB>\angle BPC+\angle CPA+\angle APB.$$

4. Find the smallest natural number n for which there exists an n-element set $M\subset\{1,2,\dots,100\}$ with the following properties:

   (i) 1 and 100 are in M;

   (ii) for every $a\in M$, $a>1$, there exist $x,y\in M$ such that $a=x+y$.

3-rd and 4-th Grades

1. Find all natural numbers smaller than 1000 that are equal to the sum of the factorials of their digits.

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2. Suppose a polynomial p with real coefficients satisfies $p(\cos x) = p(\sin x)$ for all real x. Prove that there exists a polynomial q such that for all real numbers t,

$$p(t) = q(t^4 - t^2).$$

3. In a triangle ABC the bisectors of angles at A, B, C meet the circumcircle again at P, Q, R, respectively. Prove that

$$AP + BQ + CR > AB + BC + CA.$$

4. The entries of an $n \times n$ board are integers such that the numbers written in two neighboring cells differ by at most 1. (Two cells are neighboring if they share a side.) Prove that there exists a number which appears at least n times in the board.

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