24-th Yugoslav Federal Mathematical Competition 1983

High School
Priština, April 1983

1-st Grade

1. Find all natural number n with the property that the numbers n^3 and n^4 can be written using each of the decimal digits $0,1,\dots,9$ exactly once.
2. A table with 1983 rows is formed in the following way. In the top row the numbers $1,9,8,3$ are written in this order. Thereafter, under each number one writes the sum of the remaining numbers in that row decreased by the number itself. What is the first number in the bottom row?
3. In a triangle ABC, $CA = CB$ and $\angle ACB = 80^\circ$. A point M inside the triangle is such that $\angle MBA = 30^\circ$ and $\angle MAB = 10^\circ$. Determine $\angle AMC$.
4. We call a *dolphin* a figure which is moved on a chessboard one cell up, one cell right, or diagonally one cell down-left. Can a dolphin, starting from the cell at the lower-left corner, visit each cell of the chessboard exactly once and return to the starting cell?

2-nd Grade

1. If x,y,z are positive numbers such that $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=1$, prove that
$$(x-1)(y-1)(z-1) \geq 8.$$
2. Show that for every real number $x \geq 1/2$ there is an integer n such that
$$|x^2-n| \leq \sqrt{x-\frac{1}{4}}.$$
3. Let $ABCD$ be a given rectangle and let M be an arbitrary point on the shorter arc AB of the circumcircle of $ABCD$. The line through M parallel to BC meets segments AB and CD at P and R respectively, and the line through M parallel to AB meets BC and DA at Q and S respectively. Prove that lines PQ and RS are perpendicular and that they intersect on a diagonal of the rectangle $ABCD$.
4. A rectangle $1 \times n$ consists of n unit squares ($n \geq 4$) which are numbered 1 to n. There is a marker on each of the squares $n-2, n-1, n$. Two players play the following game: they alternately make moves, and a move consists of moving one marker to an unoccupied square numbered with a smaller number. The player who cannot make a regular move loses the game. Prove that the player who plays first has a winning strategy.

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3-rd and 4-th Grades

1. Let p and q be complex numbers. Show that the solutions of the equation $x^2 + px + q = 0$ are both of module 1 if and only if $|p| \leq 2$, $|q| = 1$ and p^2/q is a nonnegative real number.

2. A function f is defined on the set of integer and satisfies

$$f(x) = \begin{cases} x - 10, & \text{if } x > 100, \\ f(f(x+11)), & \text{if } x \leq 100. \end{cases}$$

Prove that $f(x) = 91$ for each $x \leq 100$.

3. Let P be an interior point of a triangle ABC such that $\angle PAC = \angle PBC$ and let M, L be the feet of the perpendiculars from P to AC and BC, respectively. If D is the midpoint of AB, prove that $DL = DM$.

4. A sequence of positive integers (x_n) is defined by $x_1 = 2$ and $x_{n+1} = [3x_n/2]$ for $n = 1, 2, 3, \dots$ Prove that there are infinitely many odd terms and infinitely many even terms in the sequence (x_n).