23-rd Yugoslav Federal Mathematical Competition 1982

High School

Sarajevo, April 1982

1-st Grade

1. If a,b,c,d are natural numbers satisfying $(a+b)^2+a=(c+d)^2+c$, show that $a=c$ and $b=d$.

2. Each of n bulbs ($n\geq 13$) is assigned a switch. It is allowed to change states of any 13 bulbs at a moment. Initially some bulbs are *on* and some are *off*.

   (a) Is it possible to take off all the bulbs?

   (b) At least how many steps are necessary to do so, if $n=111$ and if initially all the bulbs are lit?

3. Six discs are given in a plane so that the center of any of them is not contained in the union of the remaining five discs. Show that the intersection of these discs is empty.

4. In a convex quadriulateral $ABCD$, lines AB and CD meet at point W, and X,Y are the midpoints of the diagonals AC and BD. Prove that $S_{ABCD}=4S_{XYW}$.

2-nd Grade

1. Determine the set S of the smallest possible cardinality with the following properties:

   (a) $S\in\{0,1,2,3,\dots\}$;

   (b) $1981\in S$;

   (c) if $x,y,z\in S$, then the remainder of $x+y+z$ upon division by 1982 is also in S.

2. A line l is drawn through the orthocenter of a triangle ABC. Prove that the lines symmetric to l with respect to the sides of the triangle intersect on the circumcircle of triangle ABC.

3. Let $a_1<a_2<\cdots<a_k\leq n$ be natural numbers such that $k>\left[\frac{n+1}{2}\right]$. Show that there exist indices i and r such that $a_i+a_1=a_r$.

4. Determine real numbers a,b such that for any two real numbers u,v,

$$\max_{0\leq x\leq 1}|x^2-ux-v|\geq\max_{0\leq x\leq 1}|x^2-ax-b|.$$

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3-rd and 4-th Grades

1. Find all polynomials $p(x)$ with integer coefficients such that for all real x,

$$16p(x^2) = p(2x)^2.$$

2. Prove that

$$\sqrt[44]{\tan 1^\circ \tan 2^\circ \cdots \tan 44^\circ} < \tan 22^\circ 30' < \frac{1}{44}(\tan 1^\circ + \tan 2^\circ + \cdots + \tan 44^\circ).$$

3. Let $\mathscr{M}$ be the set of points with integer coordinates in the cartesian plane and $\mathscr{S}$ be a subset of $\mathscr{M}$. We say that a mapping $f : \mathscr{S} \to \mathscr{S}$ is *neighboring* if it is a bijection and, for each $P \in \mathscr{S}$, points P and $f(P)$ are neighboring. If there exists a neighboring mapping $f : \mathscr{S} \to \mathscr{S}$, prove that there exists a neighboring mapping $g : \mathscr{S} \to \mathscr{S}$ such that $g(g(P)) = P$ for each $P \in \mathscr{S}$.

4. Prove that there is exactly one quadruple (x, y, z, t) of natural numbers with the following properties:

   (a) $1 < x < y < z < t$;

   (b) each of these numbers divides the product of the remaining three numbers increased by 1.