20-th Yugoslav Federal Mathematical Competition 1979

High School

Novi Sad, 1979

1-st Grade

1. Explain how to determine the real numbers $x_1 < x_2 < \cdots < x_5$, given their pairwise sums $S_1 < S_2 < \cdots < S_{10}$.
2. A (convex) heptagon with three angles equal to 120° is inscribed in circle k. Show that at least two sides of this heptagon are equal.
3. Is it possible to place several disjoint circles with the sum of radii 1979 within a unit circle?
4. For which positive integers n is the sum of digits of $n!$ equal to 9?

2-nd Grade

1. Points P and M on respective sides DC and BC of a square $ABCD$ are such that PM is tangent to the circle with center A and radius AB. Diagonal BD meets PA at Q and MA at N. Show that the points P,Q,M,N,C lie on a circle.
2. If $x > y \geq 0$, prove the inequality
$$x + \frac{4}{(x-y)(y+1)^2} \geq 3.$$
3. Find all representations of number 2001 as a sum of 1979 positive squares.
4. We are given $m+n$ ballots arranged in a line, where m and n are coprime positive integers. In each step we choose the leftmost m ballots and move them to the right of the remaining n, without changing their order. Show that after several steps one can bring the first ballot to an arbitrary position in the line.

3-rd Grade

1. Consider two polynomials with complex coefficients:
$$P(x) = x^n + a_1x^{n-1} + \cdots + a_n \text{ with the zeros } x_1, x_2, \dots, x_n, \quad \text{and}$$
$$x^n + b_1x^{n-1} + \cdots + b_n \text{ with the zeros } x_1^2, x_2^2, \dots, x_n^2.$$
Prove that if both $a_1 + a_3 + a_5 + \cdots$ and $a_2 + a_4 + a_6 + \cdots$ are real, then so is $b_1 + b_2 + \cdots + b_n$.

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2. A regular tetrahedron and a regular quadrilateral pyramid, both with all edges of length a, are given. Cut these figures into pieces with which one can build a cube.

3. Let $z_1, z_2, \ldots, z_n$ be complex numbers. Prove that one can select indices $1 \le i_1 < i_2 < \cdots < i_k \le n$ such that

$$|z_{i_1} + z_{i_2} + \cdots + z_{i_k}| \ge \frac{1}{4\sqrt{2}} \left(|z_1| + |z_2| + \cdots + |z_n|\right).$$

4. On each black square in the first six rows of a chessboard there is a pawn. In each move, some pawn jumps over a neighboring pawn onto a free square, and the jumped over pawn is removed from the chessboard. Is it possible that, after several moves, only one pawn remains on the chessboard?

4-th Grade

1. Prove that there are no positive integers n and $p > 5$ such that $(p-1)! + 1 = p^n$.

2. Do there exist positive numbers a, b such that

   (a) $a, b \notin \mathbb{Q}$ and $a^b \in \mathbb{Q}$?

   (b) $a, b, a^b \notin \mathbb{Q}$?

   (c) $a \in \mathbb{Q}$ and $b, a^b \notin \mathbb{Q}$?

3. For points A, B, C on a circle, P denotes the area of triangle ABC and P_1 that of the triangle enclosed by the tangents at A, B, and C. Find the limit of P_1/P when point A is fixed and points B and C approach point A along the circle so that $B \ne C$.

4. *Problem 3 for Grade 3.*