19-th Yugoslav Federal Mathematical Competition 1978

High School
Bečići, 1978

1-st Grade

1. Determine the value of the expression
$$S = \frac{1}{1+x+xy} + \frac{1}{1+y+yz} + \frac{1}{1+z+zx},$$
if the real numbers x, y, z satisfy $xyz = 1$.

2. Find all integers that are 33 times bigger than the sum of their digits.

3. Let AA_1, BB_1, CC_1 parallel chords of a circle. The points A', B', C' are symmetric to A_1, B_1, C_1, respectively with respect to the midpoints of BC, CA, and AB. Prove that A', B', and C' are collinear.

4. The board 9×10 was initially covered by 2×1 dominoes. Prove that it is impossible to cover the board using the same dominoes in such a way that all dominoes that were vertical in the first covering become horizontal in the second, and all dominoes that were horizontal in the first covering become vertical in the second.

2-nd Grade

1. Do there exist real numbers a, b, c, d such that all of the following conditions are satisfied:
   (i) The equation $ax^2 + bdx + c = 0$ has different real solutions x_1 and x_2,
   (ii) The equation $bx^2 + cdx + a = 0$ has different real solutions x_2 and x_3,
   (iii) The equation $cx^2 + adx + b = 0$ has different real solutions x_3 and x_1?

2. Let S be the subset of the set of real numbers such that the following conditions are satisfied:
   (i) $\mathbb{Z} \subseteq S$,
   (ii) $\sqrt{2} + \sqrt{3} \in S$,
   (iii) If $x, y \in S$ then $x + y \in S$ and $xy \in S$.

   Prove that $\frac{1}{\sqrt{2}+\sqrt{3}} \in S$.

3. Given a quadrilateral $ABCD$, denote by E the point of DB such that $AE \| DC$. Let F be the point of AC such that $DF \| AB$. Prove that $EF \| BC$.

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
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4. Let $A_1, A_2, \ldots, A_n$ the points of a plane such that the biggest and the smallest of the distances A_iA_j $(i \neq j)$ are equal to 1 and d, respectively. Prove that $d < \frac{2}{\sqrt{n}-1}$.

3-rd Grade

1. Find all natural numbers n for which there exists a polynomial $P_n(x)$ of degree n with integral coefficients, n different integral zeroes, and $P_n(0) = 0$.

2. Prove that an integer $r > 2$ is composite, if and only if at least one of the following two statements hold:

   (i) For some $s \in \{2, 3, \ldots\}$ the following equality holds $r = 2^s$;

   (ii) For some $u, v \in \{3, 4, \ldots\}$ such that $u \leq v$ the following equality holds: $r = \frac{u}{2}(2v - u + 1)$.

3. Let T be the centroid, and O arbitrary point inside $\triangle ABC$. If A_1, B_1, C_1 are intersections of OT with BC, CA, and AB, respectively, prove that

$$OA_1 \cdot OB_1 \cdot OC_1 \leq TA_1 \cdot TB_1 \cdot TC_1.$$

4. Points $A_1, A_2, \ldots, A_n$ are given in the plane in such a way that $A_iA_j \geq 1$ for all $i \neq j$ and $i, j \in \{1, 2, \ldots, n\}$. Prove that the number of segments A_iA_j of length exactly one can not exceed $3n$.

4-th Grade

1. Let n be a natural number. Denote by p_k the number of non-negative integer solutions to $kx + (k+1)y = n - k + 1$. Determine the sum $p_1 + p_2 + \cdots + p_{n+1}$.

2. Let $a + (n-1)d$, $n = 1, 2, \ldots$ be an arithmetic sequence of difference $d > 0$. Prove that a/d is a rational number, if and only if the sequence $a + (n-1)d$ has a geometric subsequence.

3. Assume that the set $P \subseteq \mathbb{N}$ satisfies the following two conditions:

   (i) $a \in P$ and $b \in P$ imply that $a + b \in P$;

   (ii) $(\forall q \in \mathbb{N}) q > 1 \Rightarrow (\exists c \in P) c \equiv 0 \pmod{q}$.

   Prove that the set $N \setminus P$ is finite.

4. Let $a \geq 3$ and let $P_n(x)$ be a polynomial of degree n with real coefficients. Prove that

$$\max_{0 \leq i \leq n+1} \left| a^i - P_n(i) \right| \geq 1.$$