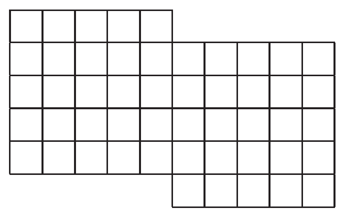
18-th Yugoslav Federal Mathematical Competition 1977

High School
Velenje, 1977

1-st Grade

1. Find all integer solutions to $p(x+y)=xy$, where p is a given prime number.
2. Let a, b, and c be natural numbers such that $a^2+b^2=c^2$. Prove that abc is divisible by 60.
3. Let P, Q, and R be points on the sides of the square $ABCD$ that divide its perimeter in three equal parts. Prove that the sum of the lengths of the segments PO, QO, and RO minimal possible if one of the points P, Q, R is midpoint of a side of the square. (The point O is the center of the square)
4. Two 5×5 tables are arranged as in the picture.



   Is it possible to cover the obtained figure using the 2×1 dominoes (one dominoe covers two adjacent squares)?

2-nd Grade

1. Solve the following equation in the set of real numbers:
$$x=\sqrt{x-\frac{1}{x}}+\sqrt{1-\frac{1}{x}}.$$
2. The points P, Q, and R are given on the sides AB, BC, and CA of $\triangle ABC$ such that
$$AP=\lambda AB,\ BQ=\lambda BC,\ CR=\lambda CA,\ \left(\frac{1}{2}\le\lambda\le 1\right).$$
   Prove that the perimeter of $\triangle PQR$ is not bigger than the perimeter of $\triangle ABC$ multiplied by λ.
3. Given 20 positive integers $a_1, a_2, \dots, a_{20}$ such that
$$a_1<a_2<\dots<a_{20}<70,$$
   prove that among the differences a_j-a_k ($j>k$) there are at least four that are equal to each other.

The IMO Compendium Group,
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4. Prove that for every integer $n > 1$ the following inequalities hold:

$$\sqrt{n} < \frac{3}{2} \cdot \frac{5}{4} \cdot \frac{7}{6} \cdots \frac{2n-1}{2n-2} < \sqrt{2n}.$$

3-rd Grade

1. Let k be a positive integer and $a_1, a_2, \dots, a_k$ positive real numbers smaller than 1. Prove the inequality:

$$\sqrt{a_1^2+(1-a^2)^2}+\sqrt{a_2^2+(1-a_3)^2}+\cdots+$$
$$+\sqrt{a_{2k-1}^2+(1-a_{2k})^2}+\sqrt{a_{2k}^2+(1-a_1)^2} \quad \geq \quad k\sqrt{2}.$$

2. Find all positive integers whose square is equal to the fifth power of the sum of its digits (in decimal representation).

3. The incircle of a right triangle with hypothenuse c touches the sides of an acute angle at M and N. Prove that $MN < 2c\sqrt{3}/9$.

4. Let D be the set of the diagonals of a 100-gon. Does there exist a subset S of D with the following three properties:

   (i) No two of the diagonals from E have common interior points;

   (ii) For each of the vertices of the 100-gon there is an even number of diagonals originating from it;

   (iii) The diagonals from E partition the 100-gon into triangles.

4-th Grade

1. Let $n \geq 2$ and $a_j = n! + j$, for $j \in \{1,2,\dots,n\}$. Prove that for each $k \in \{1,2,\dots,n\}$ there exist at least one prime number p such that a_k is divisible by p, and none of the numbers $a_1, a_2, \dots, a_{k-1}, a_{k+1}, \dots, a_n$ is divisible by p.

2. Prove that the area of the square that is entirely inside the given triangle can't be bigger than the half of the area of the triangle.

3. In how many different ways one can express $6k$ ($k \in \mathbb{N}$) as a sum of three positive integers? The representations that differ only in the order of the summands are considered the same.

4. Given a set S of 100 points in a plane, prove that there are finitely many circles in the plane such that:

   (i) Each point of S is contain in the interior of at least one of the circles;

   (ii) The circles are disjoint and the distance between each two of them is strictly greater than 1;

   (iii) The sum of the diameters of the given circles is strictly less than 100.