Serbian Mathematical Olympiad 2008

Belgrade, April 12–13

First Day

1. Solve in integers the equation $12^x + y^4 = 2008^z$.

2. Given a triangle ABC, let D and E be the points on line AB such that $D-A-B-E$, $AD = AC$ and $BE = BC$. The bisectors of the angles at A and B meet the opposite sides of the triangle at P and Q respectively, and meet the circumcircle at M and N, respectively. The line joining A with the circumcenter of triangle BME and the line joining B with the circumcenter of triangle AND intersect at point X. Prove that $CX \perp PQ$.

3. If a,b and c are arbitrary positive numbers with $a+b+c=1$, prove the inequality
$$\frac{1}{bc+a+\frac{1}{a}} + \frac{1}{ca+b+\frac{1}{b}} + \frac{1}{ab+c+\frac{1}{c}} \leq \frac{27}{31}.$$

Second Day

4. Each point of a plane is painted in one of three colors. Show that there exists a triangle such that:
   (i) all three vertices of the triangle are of the same color;
   (ii) the radius of the circumcircle of the triangle is 2008;
   (iii) one angle of the triangle is either two or three times greater than one of the other two angles.

5. The sequence $(a_n)_{n\geq 1}$ is defined by $a_1 = 3$, $a_2 = 11$ and $a_n = 4a_{n-1} - a_{n-2}$, for $n \geq 3$. Prove that each term of this sequence is of the form $a^2 + 2b^2$ for some natural numbers a and b.

6. Let $ABCDE$ be a convex pentagon in which $AB = 1$, $\angle BAE = \angle ABC = 120^\circ$, $\angle CDE = 60^\circ$ and $\angle ADB = 30^\circ$. Prove that the area of pentagon $ABCDE$ is less than $\sqrt{3}$.

Time allowed: 270 minutes each day.
Each problem is worth 7 points.

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
www.imomath.com