

# Serbian Mathematical Olympiad 2007

Belgrade, April 2–3

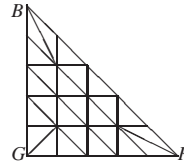
*First Day*

1. Let  $D$  be the point on side  $AC$  of a triangle  $ABC$  with  $AB < BC$  such that  $AB = BD$ . The incircle of  $\triangle ABC$  touches  $AB$  at  $K$  and  $AC$  at  $L$ , and  $J$  is the incenter of triangle  $BCD$ . Prove that  $KL$  bisects the segment  $AJ$ .

2. Triangle  $GRB$  is dissected into 25 "small" triangles as shown. All vertices of these triangles are painted in three colors so that the following conditions are satisfied:

Vertex  $G$  is painted in green, vertex  $R$  in red, and  $B$  in blue;

Each vertex on side  $GR$  is either green or red, each vertex on  $RB$  is either red or blue, and each vertex on  $GB$  is either green or blue. The vertices inside the big triangle are arbitrarily colored.



Prove that, regardless of the way of coloring, at least one of the 25 small triangles has vertices of three different colors.

3. Determine all pairs of natural numbers  $(x, n)$  that satisfy the equation

$$x^3 + 2x + 1 = 2^n.$$

*Second Day*

4. Let  $k$  be a natural number. For each function  $f : \mathbb{N} \rightarrow \mathbb{N}$  define the sequence of functions  $(f_m)_{m \geq 1}$  by  $f_1 = f$  and  $f_{m+1} = f \circ f_m$  for  $m \geq 1$ . Function  $f$  is called  $k$ -nice if  $f_k(n) = f(n)^k$  for each  $n \in \mathbb{N}$ .

- (a) For which  $k$  does there exist an injective  $k$ -nice function  $f$ ?  
(b) For which  $k$  does there exist a surjective  $k$ -nice function  $f$ ?

5. In a scalene triangle  $ABC$ ,  $AD$ ,  $BE$ ,  $CF$  are the angle bisectors ( $D \in BC$ ,  $E \in AC$ ,  $F \in AB$ ). Points  $K_a$ ,  $K_b$ ,  $K_c$  on the incircle of triangle  $ABC$  are such that  $DK_a$ ,  $EK_b$ ,  $FK_c$  are tangent to the incircle and  $K_a \notin BC$ ,  $K_b \notin AC$ ,  $K_c \notin AB$ . Let  $A_1$ ,  $B_1$ ,  $C_1$  be the midpoints of sides  $BC$ ,  $CA$ ,  $AB$ , respectively. Prove that the lines  $A_1K_a$ ,  $B_1K_b$ ,  $C_1K_c$  intersect on the incircle of triangle  $ABC$ .

6. Let  $k$  be a given natural number. Prove that for any positive numbers  $x, y, z$  with the sum 1 the following inequality holds:

$$\frac{x^{k+2}}{x^{k+1} + y^k + z^k} + \frac{y^{k+2}}{y^{k+1} + z^k + x^k} + \frac{z^{k+2}}{z^{k+1} + x^k + y^k} \geq \frac{1}{7}.$$

When does equality occur?

Time allowed: 270 minutes.  
Each problem is worth 7 points.