45-th Federal Mathematical Competition of Serbia and Montenegro 2005

High School

Budva, April 16, 2005

Time allowed 4 hours.
Each problem is worth 25 points.

1-st Grade

1. Find all positive integers n with the following property: For every positive divisor d of n, $d+1$ divides $n+1$.

2. Let ABC be an acute triangle. Circle k with diameter AB intersects AC and BC again at M and N respectively. The tangents to k at M and N meet at point P. Given that $CP = MN$, determine $\angle ACB$.

3. If x, y, z are nonnegative numbers with $x + y + z = 3$, prove that
$$\sqrt{x} + \sqrt{y} + \sqrt{z} \geq xy + yz + zx.$$

4. There are c red, p blue, and b white balls on a table. Two players A and B play a game by alternately making moves. In every move, a player takes two or three balls from the table. Player A begins. A player wins if after his/her move at least one of the three colors no longer exists among the balls remaining on the table. For which values of c, p, b does player A have a winning strategy?

2-nd Grade

1. Let A and b be positive integers and $K = \sqrt{\dfrac{a^2+b^2}{2}}$, $A = \dfrac{a+b}{2}$. If $\dfrac{K}{A}$ is a positive integer, prove that $a = b$.

2. Every square of a 3×3 board is assigned a sign $+$ or $-$. In every move, one square is selected and the signes are changed in the selected square and all the neighboring squares (two squares are neighboring if they have a common side). Is it true that, no matter how the signs were initially distributed, one can obtain a table in which all signs are $-$ after finitely many moves?

3. In a triangle ABC, D is the orthogonal projection of the incenter I onto BC. Line DI meets the incircle again at E. Line AE intersects side BC at point F. Suppose that the segment IO is parallel to BC, where O is the circumcenter of $\triangle ABC$. If R is the circumcenter and r the incenter of the triangle, prove that $EF = 2(R - 2r)$.

4. Inside a circle k of radius R some round spots are made. The area of each spot is 1. Every radius of circle k, as well as every circle concentric with k, meets no more than one spot. Prove that the total area of all the spots is less than
$$\pi\sqrt{R} + \frac{1}{2}R\sqrt{R}.$$

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3-rd and 4-th Grades

1. If x,y,z are positive numbers, prove that
$$\frac{x}{\sqrt{y+z}}+\frac{y}{\sqrt{z+x}}+\frac{z}{\sqrt{x+y}}\geq\sqrt{\frac{3}{2}(x+y+z)}.$$

2. Suppose that in a convex hexagon, each of the three lines connecting the midpoints of two opposite sides divides the hexagon into two parts of equal area. Prove that these three lines intersect in a point.

3. Determine all polynomials p with real coefficients for which $p(0)=0$ and
$$f(f(n))+n=4f(n)\quad\text{for all } n\in\mathbb{N},$$
where $f(n)=[p(n)]$.

4. On each cell of a 2005×2005 chessboard there is a marker. In each move, we are allowed to remove a marker which is a neighbor to an even number of markers (but at least one). Two markers are considered neighboring if their cells share a vertex.
   (a) Find the least number n of markers that we can end up with on the chessboard.
   (b) If we end up with this minimum number n of markers, prove that no two of them will be neighboring.