43-rd Federal Mathematical Competition of Serbia and Montenegro 2003

High School

Novi Sad, April 19, 2003

Time allowed 4 hours.
Each problem is worth 25 points.

1-st Grade

1. Find the number of solutions to the equation

$$x_1{}^4 + x_2{}^4 + \ldots + x_{10}{}^4 = 2011$$

   in the set of positive integers.

2. Given a segment AB of length 2003 in a coordinate plane, determine the maximal number of unit squares with vertices in the lattice points whose intersection with the given segment is non-empty.

3. Let a,b and c be the lengths of the edges of a triangle whose angles are $\alpha = 40^\circ$, $\beta = 60^\circ$ and $\gamma = 80^\circ$. Prove that

$$a(a+b+c) = b(b+c).$$

4. An acute angle with the vertex O and the rays Op_1 and Op_2 is given in a plane. Let k_1 be a circle with the center on Op_1 which is tangent to Op_2. Let k_2 be the circle which is tangent to both rays Op_1 and Op_2 and to the circle k_1 from outside. Find the locus of tangency points of k_1 and k_2 when center of k_1 moves along the ray Op_1.

2-nd Grade

1. Given a $\triangle ABC$ with the edges a, b and c and the area S.

   (a) Prove that there exists $\triangle A_1B_1C_1$ with the sides $\sqrt{a}$, $\sqrt{b}$ and $\sqrt{c}$.

   (b) If S_1 is the area of $\triangle A_1B_1C_1$, prove that $S_1{}^2 \geq \dfrac{S\sqrt{3}}{4}$.

2. Let $ABCD$ be a square inscribed in a circle k and P be an arbitrary point of that circle. Prove that at least one of the numbers PA, PB, PC and PD is not rational.

3. Let $ABCD$ be a rectangle. Determine the set of all points P from the region between the parallel lines AB and CD such that $\measuredangle APB = \measuredangle CPD$.

4. Let S be a subset of $\mathbb{N}$ with the following properties:

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1° Among each 2003 consecutive natural numbers there exists at least one contained in S;

2° If $n \in S$ and $n > 1$, then $\left[\frac{n}{2}\right] \in S$.

Prove that $S = \mathbb{N}$.

3-rd and 4-th Grades

1. Prove that the number $\left[(5+\sqrt{35})^{2n-1}\right]$ is divisible by 10^n for each $n \in \mathbb{N}$.

2. Let $f : [0,1] \to \mathbb{R}$ be a function with the following properties:

   1° $f(x) \geq 0$ for all $x \in [0,1]$;

   2° $f(1) = 1$,

   3° if $x_1, x_2 \in [0,1]$ and $x_1 + x_2 \leq 1$, then $f(x_1) + f(x_2) \leq f(x_1 + x_2)$.

   Prove that for all $x \in [0,1]$ it holds $f(x) \leq 2x$.

3. Given a circle k and the point P outside it, an arbitrary line s passing through P intersects k at the points A and B. Let M and N be the midpoints of the arcs determined by the points A and B and let C be the point on AB such that $PC^2 = PA \cdot PB$. Prove that $\measuredangle MCN$ doesn't depend on the choice of s.

4. Let n be an even number and S be the set of all arrays of the length n whose elements are 0 and 1. Prove that S can be partitioned into disjoint three-element subsets such that: for each three arrays $(a_i)_{i=1}^n$, $(b_i)_{i=1}^n$, $(c_i)_{i=1}^n$ which belong to the same subset and all $i \in \{1, 2, \ldots, n\}$ the number $a_i + b_i + c_i$ is divisible by 2.

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