42-th Federal Mathematical Competition of Serbia and Montenegro 2002

High School

Bečići, April 20, 2002

Time allowed 4 *hours.*
Each problem is worth 25 *points.*

1-st Grade

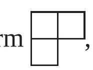
1. Determine all real numbers x such that

$$\frac{2002[x]}{[-x]+x} > \frac{[2x]}{x-[1+x]}.$$

2. Let O be a point inside a triangle ABC and let the lines AO, BO, and CO meet sides BC, CA, and AB at points A_1, B_1, and C_1, respectively. If AA_1 is the longest among the segments AA_1, BB_1, CC_1, prove that

$$OA_1 + OB_1 + OC_1 \leq AA_1.$$

3. Find all pairs (n,k) of positive integers such that $\binom{n}{k} = 2002$.

4. Is it possible to cut a rectangle 2001×2003 into pieces of the form , each consisting of three unit squares?

2-nd Grade

1. Real numbers x, y, z satisfy the inequalities

$$x^2 \leq y+z, \quad y^2 \leq z+x, \quad z^2 \leq x+y.$$

Find the minimum and maximum possible values of z.

2. Points $A_0, A_1, \dots, A_{2k}$, in this order, divide a circumference into $2k+1$ equal arcs. Point A_0 is connected by chords to all the other points. These $2k$ chords divide the interior of the circle into $2k+1$ parts. These parts are alternately painted red and blue so that there are $k+1$ red and k blue parts. Show that the blue area is larger than the red area.

3. Let m and n be positive integers. Prove that the number $2^n - 1$ is divisible by $(2^m - 1)^2$ if and only if n is divisible by $m(2^m - 1)$.

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4. Each of the 15 coaches ranked the 50 selected football players on the places from 1 to 50. For each football player, the highest and lowest obtained ranks differ by at most 5. For each of the players, the sum of the ranks he obtained is computed, and the sums are denoted by $S_1 \leq S_2 \leq \cdots \leq S_{50}$. Find the largest possible value of S_1.

3-rd and 4-th Grades

1. For any positive numbers a,b,c and natural numbers n,k prove the inequality

$$\frac{a^{n+k}}{b^n}+\frac{b^{n+k}}{c^n}+\frac{c^{n+k}}{a^n} \geq a^k+b^k+c^k.$$

2. The (Fibonacci) sequence f_n is defined by $f_1 = f_2 = 1$ and $f_{n+2} = f_{n+1} + f_n$ for $n \geq 1$. Prove that the area of the triangle with the sides $\sqrt{f_{2n+1}}$, $\sqrt{f_{2n+2}}$ and $\sqrt{f_{2n+3}}$ is equal to $\frac{1}{2}$.

3. Let $ABCD$ be a rhombus with $\angle BAD = 60^\circ$. Points S and R inside the triangles ABD and DBC, respectively, are chosen such that

$$\angle SBR = \angle RDS = 60^\circ.$$

   Prove that $SR^2 \geq AS \cdot CR$.

4. Is there a positive integer k such that none of the digits $3,4,5,6$ occurs in the decimal representation of the number $2002! \cdot k$?