41-st Yugoslav Federal Mathematical Competition 2001

High School

Kragujevac, April 21, 2001

Time allowed 4 hours.
Each problem is worth 25 points.

1-st Grade

1. Let $ABCD$ and $A_1B_1C_1D_1$ be convex quadrangles in a plane, such that $AB = A_1B_1$, $BC = B_1C_1$, $CD = C_1D_1$ and $DA = D_1A_1$. Given that diagonals AC and BD are perpendicular to each other, prove that the same holds for diagonals A_1C_1 and B_1D_1.

2. Given are 5 segments, such that from any three of them one can form a triangle. Prove that from some three of them one can form an acute-angled triangle.

3. Let $p_1, p_2, \dots, p_n$ $(n \geq 3)$ be the smallest n prime numbers. Prove that

$$\frac{1}{p_1^2} + \frac{1}{p_2^2} + \cdots + \frac{1}{p_n^2} + \frac{1}{p_1 p_2 \cdots p_n} < \frac{1}{2}.$$

4. There are n coins in the pile. Two players play a game by alternately performing a move. A move consists of taking 5, 7 or 11 coins away from the pile. The player unable to perform a move loses the game. Which player - the one playing first or second - has the winning strategy if:

   (a) $n = 2001$;

   (b) $n = 5000$?

2-nd Grade

1. Let $S = \{x^2 + 2y^2 \mid x, y \in \mathbb{Z}\}$. If a is an integer with the property that $3a$ belongs to S, prove that then a belongs to S as well.

2. Vertices of a square $ABCD$ of side $25/4$ lie on a sphere. Parallel lines passing through points A, B, C and D intersect the sphere at points A_1, B_1, C_1 and D_1, respectively. Given that $AA_1 = 2$, $BB_1 = 10$, $CC_1 = 6$, determine the length of the segment DD_1.

3. Determine all positive integers n for which there is a coloring of all points in space so that each of the following conditions is satisifed:

   (i) Each point is painted in exactly one color.

   (ii) Exactly n colors are used.

   (iii) Each line is painted in at most two different colors.

The IMO Compendium Group,
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4. Let S be the set of all $n-$tuples $(x_1, x_2, \dots, x_n)$ of real numbers, with the property that among the numbers $x_1, \frac{x_1+x_2}{2}, \dots, \frac{x_1+x_2+\ldots+x_n}{n}$ the least is equal to 0, and the greatest is equal to 1. Determine

$$\max_{(x_1,\dots,x_n)\in S} \max_{1\le i,j\le n} (x_i - x_j) \quad \text{and} \quad \min_{(x_1,\dots,x_n)\in S} \max_{1\le i,j\le n} (x_i - x_j).$$

3-rd and 4-th Grades

1. Find all solutions of the equation $x^y + y = y^x + x$ in the positive integers.

2. Let $x_1, x_2, \dots, x_{2001}$ be positive numbers such that

$$x_i^2 \ge x_1^2 + \frac{x_2^2}{2^3} + \frac{x_3^2}{3^3} + \cdots + \frac{x_{i-1}^2}{(i-1)^3} \quad \text{for } 2 \le i \le 2001.$$

Prove that $\sum_{i=2}^{2001} \frac{x_i}{x_1 + x_2 + \cdots + x_{i-1}} > 1.999$.

3. Let k be a positive integer and N_k be the number of sequences of length 2001, all members of which are elements of the set $\{0, 1, 2, \dots, 2k+1\}$, and the number of zeroes among these is odd. Find the greatest power of 2 which divides N_k.

4. Parallelogram $ABCD$ is the base of a pyramid $SABCD$. Planes determined by triangles ASC and BSD are mutually perpendicular. Find the area of the side ASD, if areas of sides ASB, BSC and CSD are equal to x, y and z, respectively.