Vietnamese IMO Team Selection Test 1998

First Day – May 14

1. Suppose that a function $f : \mathbb{R} \to \mathbb{R}$ is such that, for every $c > 0$, there is a polynomial $P_c(x)$ satisfying
$$|f(x) - P_c(x)| \le cx^{1998} \qquad \text{for all } x \in \mathbb{R}.$$
Prove that f is itself a polynomial.

2. Let be given a circle A with radius R and a circle B passing through the center of A and touching internally with A. Let $\mathscr{H}$ be the family of circles C touching B externally and A internally. Let $n > 1$ be an integer and C, C' be two circles in $\mathscr{H}$ whose bends (i.e. the reciprocals of radii) are p and p'. Prove that there is a chain of circles $C = C_1, C_2, \dots, C_n = C'$ in $\mathscr{H}$ such that C_i touches C_{i+1} for all i, if and only if
$$(p - p')^2 = (n-1)^2(2p + 2p' - (n-1)^2 - 8).$$

3. Let $m > 3$ be an integer and $p_1, p_2, \dots, p_n$ be all prime numbers not exceeding m. Prove that
$$\sum_{k=1}^{n} \left(\frac{1}{p_k} + \frac{1}{p_k^2} \right) > \ln \ln m.$$

Second Day – May 15

4. Find all monic polynomials P with integer coefficients with the property that $P(a)$ is an integer for infinitely many irrational numbers a.

5. Led d be a positive divisor of $1998^{1998} + 5$. Prove that d can be written in the form $d = 2x^2 + 2xy + 3y^2$ $(x, y \in \mathbb{Z})$ if and only if $d \equiv 3$ or $d \equiv 7 \pmod{20}$.

6. Suppose that a group of $n \ge 10$ persons has the following properties:

   (1) Each person is acquainted to at least $[\frac{n+2}{3}]$ others;

   (2) For any two persons A and B who are not acquainted, there is a chain of persons $A = A_0, A_1, \dots, A_k = B$ such that A_i is acquainted to A_{i+1} for each i;

   (3) The persons cannot be arranged in a line so that any two adjacent persons are acquainted.

   Prove that this group can be partitioned into two groups such that

   (i) the persons in one group can sit around a table so that any two adjacent persons are acquainted;

   (ii) no the persons in the other group are acquainted.

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