Vietnamese IMO Team Selection Test 1995

First Day – May 5

1. Let be given a triangle ABC with the lengths of sides BC, CA, AB equal to a, b, c. Distinct points $A_1, A_2, B_1, B_2, C_1, C_2$ not coinciding with A, B, C are chosen so that for some real numbers α, β, γ,

$$\overrightarrow{A_1A_2} = \frac{\alpha}{a}\overrightarrow{BC}, \quad \overrightarrow{B_1B_2} = \frac{\beta}{b}\overrightarrow{CA}, \quad \overrightarrow{C_1C_2} = \frac{\gamma}{c}\overrightarrow{AB}.$$

   Let d_1, d_b, d_c be respectively the radical axes of the circumcircles of the pairs of triangles AB_1C_1 and AB_2C_2; BC_1A_1 and BC_2A_2; CA_1B_1 and CA_2B_2. Prove that d_a, d_b and d_c are concurrent if and only if $\alpha a + \beta b + \gamma c = 0$.

2. Find all integers k such that the polynomial

$$P(x) = x^{n+1} + kx^n - 870x^2 + 1945x + 1995$$

   is reducible over $\mathbb{Z}[x]$ for infinitely many integers $n \geq 3$.

3. Find all integers a, b, n greater than 1 which satisfy

$$(a^3 + b^3)^n = 4(ab)^{1995}.$$

Second Day – May 6

4. A graph has n vertices and $\dfrac{n^2 - 3n + 4}{2}$ edges. There is an edge such that, after removing it, the graph becomes unconnected. Find the greatest possible length k of a circuit in such a graph.

5. For any nonnegative integer n, let $f(n)$ be the greatest integer such that $2^{f(n)} \mid n + 1$. A pair (n, p) of nonnegative integers is called *nice* if $2^{f(n)} > p$. Find all triples (n, p, q) of nonnegative integers such that the pairs (n, p), (p, q) and $(n + p + q, n)$ are all nice.

6. Consider the function $f(x) = \dfrac{2x^3 - 3}{3x^2 - 3}$.

   (a) Prove that there is a continuous function $g : \mathbb{R} \to \mathbb{R}$ satisfying $f(g(x)) = x$ and $g(x) > x$ for all real x.

   (b) Show that there exists a real number $a > 1$ such that the sequence $a, f(a), f(f(a)), \ldots$ is periodic with the smallest period 1995.

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