Vietnamese IMO Team Selection Test 1994

First Day – May 18

1. A parallelogram $ABCD$ is given. Let E and F be points on the sides BC and CD respectively such that the triangles ABE and BCF have the same area. The diagonal BD intersects AE at M and AF at N.
   (a) Prove that there exists a triangle with sides equal to BM, MN, ND.
   (b) When E and F vary so that the length of MN decreases, prove that the circumradius of the triangle from (a) also decreases.

2. For a given positive integer N, consider the equation in x, y, z, t:
$$x^2 + y^2 + z^2 + t^2 = N(xyzt + 1).$$
Prove that this equation has a solution in positive integers for infinitely many values of N. Also prove that the considered equation has no solutions in positive integers if $N = 4^k(8m+7)$ for some nonnegative integers k and m.

3. Let $P(x)$ be a polynomial of degree 4 having four positive roots. Prove that the equation
$$\frac{1-4x}{x^2}P(x) + \left(1 - \frac{1-4x}{x^2}\right)P'(x) - P''(x) = 0$$
also has four positive roots.

Second Day – May 19

4. Let M be a point in the plane of an equilateral triangle ABC and let A', B', C' be respectively symmetric to A, B, C with respect to M.
   (a) Prove that there is a unique point P equidistant from A and B', from B and C', and from C and A'.
   (b) Let D be the midpoint of AB. For a point $M \neq D$, let the lines DM and AP meet at N. Prove that, when M varies, the circumcircle of $\triangle MNP$ passes through a fixed point.

5. Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying for all real x,
$$f\left(\sqrt{2}x\right) + f\left((4+3\sqrt{2})x\right) = 2f\left((2+\sqrt{2})x\right).$$

6. Evaluate
$$T = \sum \frac{1}{n_1!n_2!\cdots n_{1994}!(n_2 + 2n_3 + \cdots + 1993n_{1994})!},$$
where the sum is taken over all 1994-tuples $(n_1, \dots, n_{1994})$ of natural numbers satisfying $n_1 + 2n_2 + \cdots + 1994n_{1994} = 1994$.

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