Vietnamese IMO Team Selection Test 1993

First Day – May 4

1. Three kinds of tiles are given:

   kind 1    kind 2    kind 3

A rectangle $1993 \times 2000$ is tiled with $m$ tiles of kind 1, $n$ tiles of kind 2 and $p$ kinds of kind 3. Find the maximum possible value of $n + p$.

2. A sequence $(a_n)$ is defined by $a_1 = 1$ and $a_{n+1} = a_n + a_n^{-1/2}$. Find all real numbers $\alpha$ such that the sequence $a^{\alpha}_n/n$ converges to a nonzero limit.

3. If $x_1, x_2, x_3, x_4$ are real numbers with $\frac{1}{2} \leq x_1^2 + x_2^2 + x_3^2 + x_4^2 \leq 1$, find the maximum and minimum values of

   \[ A = (-2x_1 + x_2)^2 + (x_1 - 2x_2 + x_3)^2 + (x_2 - 2x_3 + x_4)^2 + (x_3 - 2x_4)^2. \]

Second Day – May 5

4. Let $H, O, I$ be respectively the orthocenter, circumcenter and incenter of a triangle $ABC$. Prove that $2IO \geq OH$. When does equality hold?

5. Let $\varphi(n)$ denote the Euler function. Find all integers $k > 1$ with the following property: for any positive integer $a$, the sequence $x_n$ defined by $x_0 = a$ and $x_{n+1} = k\varphi(x_n)$ for $n \geq 0$ is bounded.

6. Find the largest $n$ satisfying the following condition: There exists a graph with $n$ vertices, each vertex having degree at most 4, such that any two vertices $A$ and $B$ are either adjacent or there is another vertex $C$ which is adjacent to both $A$ and $B$. 