Vietnamese IMO Team Selection Test 1991

First Day

1. Consider all sets S consisting of n points in the plane that satisfy:

   (i) The distance between any two points of S is at most 1;

   (ii) Every point $A \in S$ has exactly two *neighbors*, where two points are said to be neighbors if they are on the distance 1 apart.

   (iii) If A and B are any two points in S, A', A'' the two neighbors of A, and B', B'' the two neighbors of B, then $\angle A'AA'' = \angle B'BB''$.

   Is there such a set S if (a) $n = 1991$; (b) $n = 2000$?

2. A non-constant and monotone sequence $a_1, a_2, \dots, a_n$ of $n > 2$ positive real numbers and real numbers x, y satisfying $\dfrac{x}{y} \geq \dfrac{a_1 - a_2}{a_1 - a_n}$ are given. Prove that
$$\sum_{k=1}^{n} \frac{a_k}{a_{k+1}x + a_{k+2}y} \geq \frac{n}{x+y},$$
   where $a_{n+i} = a_i$.

3. The sequence (x_n) is defined by $(x_1, x_2, x_3, x_4) = (1, 9, 9, 1)$ and
$$x_{n+4} = \sqrt[4]{x_n x_{n+1} x_{n+2} x_{n+3}} \quad \text{for } n \geq 1.$$
   Prove that this sequence converges and find its limit.

Second Day

4. For each tetrahedron T whose all faces are right triangles and whose edge lengths do not exceed 1, define $\sigma(T) = S_1^2 + S_2^2 + S_3^2 + S_4^2$, where $S_1, \dots, S_4$ are the areas of the faces of T. Find the maximum value of $\sigma(T)$.

5. Let n be a positive integer and let $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$, where p_i are distinct primes and a_i positive integers. Put
$$f(n) = \begin{cases} 1 & \text{if } n = 1, \\ 1 + a_1 p_1 + \cdots + a_k p_k & \text{if } n > 1. \end{cases}$$
   For every natural number s, define $f_s(n) = f(\dots f(n) \dots)$, i.e. f applied s times. Prove that for every $\beta \in \mathbb{N}$ there exists s_0 such that the sum $f_s(\beta) + f_{s-1}(\beta)$ is constant for $s > s_0$.

6. Let X be a set of $2n$ distinct real numbers ($n \geq 3$). Consider a set K of pairs of elements of X satisfying the following conditions:

(i) If $(x,y) \in K$ then $(y,x) \notin K$.

(ii) Each element x of X belongs to at most 19 pairs in K.

Prove that set X can be partitioned into five nonempty subsets X_1, X_2, X_3, X_4, X_5 so that there are at most $3n$ pairs $(x,y) \in K$ with x, y belonging to the same X_i.