Vietnamese IMO Team Selection Test 1985

First Day

1. The sequence \((x_n)\) of real numbers is defined by \(x_1 = \frac{29}{10}\) and \(x_{n+1} = \frac{x_n}{\sqrt{x_n^2 - 1}} + \sqrt{3}, \) for \(n \geq 1.\)

Find a real number \(a\) (if it exists) such that \(x_{2k-1} > a > x_{2k}\) for all \(k \in \mathbb{N}.\)

2. Let \(ABC\) be a triangle with \(AB = AC.\) A ray \(Ax\) is constructed in space such that the three planar angles of the trihedral angle \(ABCx\) at its vertex \(A\) are equal. If a point \(S\) moves on \(Ax,\) find the locus of the incenter of \(\triangle SBC.\)

3. Does there exist a triangle \(ABC\) satisfying the following two conditions:
   (i) \(\sin^2 A + \sin^2 B + \sin^2 C = \cot A + \cot B + \cot C;\)
   (ii) \(S \geq a^2 - (b - c)^2,\) where \(S\) is the area of the triangle?

Second Day

4. A convex polygon \(A_1A_2 \ldots A_n\) is inscribed in a circle with center \(O\) and radius \(R\) so that \(O\) lies inside the polygon. Let the inradii of triangles \(A_1A_2A_3, A_1A_3A_4, \ldots, A_1A_n-1A_n\) be denoted by \(r_1, r_2, \ldots, r_{n-2}.\) Show that \(r_1 + r_2 + \cdots + r_{n-2} \leq R \left( n \cos \frac{\pi}{n} - n + 2 \right).\)

5. Find all real values of \(a\) for which the equation

\[
\left( a - 3x^2 + \cos \frac{9\pi x}{2} \right) \sqrt{3 - ax} = 0
\]

has an odd number of solutions in the interval \([-1, 5].\)

6. Suppose a function \(f : \mathbb{R} \to \mathbb{R}\) satisfies \(f(f(x)) = -x\) for every \(x.\) Prove that \(f\) has infinitely many points of discontinuity.