Vietnamese IMO Team Selection Test 2008

First Day

1. Given an angle xOy in a plane, let M be a mobile point on ray Ox and N a mobile point on ray Oy. Let d be the external angle bisector of $\angle xOy$ and let I be the intersection of d with the perpendicular bisector of MN. Let P and Q be two points that lie on d such that $IP = IQ = IM = IN$, and let $K = MQ \cap NP$.

   (a) Prove that K always lie on a fixed line.

   (b) Let d_1 be the line perpendicular to IM at M and let d_2 be the line perpendicular to IN at N. Assume that d intersects d_1 and d_2 at E and F respectively. Prove that EN, FM, and OK pass through a point.

2. Find all values of the positive integer m such that there exist polynomials $P(x)$, $Q(x)$, $R(x,y)$ with real coefficients satisfying: For every real numbers a and b such that $a^m - b^2 = 0$ the following relations hold: $P(R(a,b)) = a$ and $Q(R(a,b)) = b$.

3. Given an integer $n > 3$, let $T = \{1, 2, \dots, n\}$. A subset S of T is called a *nice* set if it satisfies: There exists a positive integer c which is not greater than $n/2$ such that $|s_1 - s_2| \neq c$ for every pair of arbitrary elements $s_1, s_2 \in S$. How many nice sets are there?

Second Day

1. Let m and n be positive integers. Prove that $6m | (2m+3)^n + 1$ if and only if $4m | 3^n + 1$.

2. Let k be a positive real number. Let O be the circumcenter of the acute-angled and non-isosceles triangle ABC. Let L, M, N be the points on the internal angle bisectors AD, BE, and CF such that $\frac{AL}{AD} = \frac{BM}{BE} = \frac{CN}{CF} = k$. Denote by k_1, k_2, and k_3 the following three circles: Circle through L that touches OA at A; the circle through M that is tangent to OB at B, and the circle through N that is tangent to OC at C.

   (a) Prove that when $k = \frac{1}{2}$ the three circles k_1, k_2, and k_3 have exactly two common points.

   (b) Find all values of k for which these three circles have exactly two common points.

3. Each of the numbers from the set $M = \{1, 2, \dots, 2008\}$ is painted in one of the three colors: blue, yellow, or red. Each color is used at least once. Define the following two sets:

$$\begin{aligned} S_1 &= \big\{(x,y,z) \in M^3 \ : \ x,y,z \text{ have the same color and } 2008 \,|\, (x+y+z)\big\} \\ S_2 &= \big\{(x,y,z) \in M^3 \ : \ x,y,z \text{ have three pairwise different colors} \\ &\qquad \text{and } 2008 \,|\, (x+y+z)\big\}. \end{aligned}$$

Prove that $2|S_1| > |S_2|$.

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