Vietnamese IMO Team Selection Test 2008

First Day

1. Given an angle \(xOy\) in a plane, let \(M\) be a mobile point on ray \(Ox\) and \(N\) a mobile point on ray \(Oy\). Let \(d\) be the external angle bisector of \(\angle xOy\) and let \(I\) be the intersection of \(d\) with the perpendicular bisector of \(MN\). Let \(P\) and \(Q\) be two points that lie on \(d\) such that \(IP = IQ = IM = IN\), and let \(K = MQ \cap NP\).

(a) Prove that \(K\) always lie on a fixed line.

(b) Let \(d_1\) be the line perpendicular to \(IM\) at \(M\) and let \(d_2\) be the line perpendicular to \(IN\) at \(N\). Assume that \(d\) intersects \(d_1\) and \(d_2\) at \(E\) and \(F\) respectively. Prove that \(EN, FM,\) and \(OK\) pass through a point.

2. Find all values of the positive integer \(m\) such that there exist polynomials \(P(x), Q(x), R(x,y)\) with real coefficients satisfying: For every real numbers \(a\) and \(b\) such that \(a^m - b^2 = 0\) the following relations hold: \(P(R(a,b)) = a\) and \(Q(R(a,b)) = b\).

3. Given an integer \(n > 3\), let \(T = \{1, 2, \ldots, n\}\). A subset \(S\) of \(T\) is called a nice set if it satisfies: There exists a positive integer \(c\) which is not greater than \(n/2\) such that \(|s_1 - s_2| \neq c\) for every pair of arbitrary elements \(s_1, s_2 \in S\). How many nice sets are there?

Second Day

1. Let \(m\) and \(n\) be positive integers. Prove that \(6m|(2m + 3)^n + 1\) if and only if \(4m|3^n + 1\).

2. Let \(k\) be a positive real number. Let \(O\) be the circumcenter of the acute-angled and non-isosceles triangle \(ABC\). Let \(L, M, N\) be the points on the internal angle bisectors \(AD, BE,\) and \(CF\) such that \(\frac{AL}{AD} = \frac{BM}{BE} = \frac{CN}{CF} = k\). Denote by \(k_1, k_2,\) and \(k_3\) the following three circles: Circle through \(L\) that touches \(OA\) at \(A\); the circle through \(M\) that is tangent to \(OB\) at \(B\), and the circle through \(N\) that is tangent to \(OC\) at \(C\).

(a) Prove that when \(k = \frac{1}{2}\) the three circles \(k_1, k_2,\) and \(k_3\) have exactly two common points.

(b) Find all values of \(k\) for which these three circles have exactly two common points.

3. Each of the numbers from the set \(M = \{1, 2, \ldots, 2008\}\) is painted in one of the three colors: blue, yellow, or red. Each color is used at least once. Define the following two sets:

\[
S_1 = \{(x,y,z) \in M^3 : x,y,z \text{ have the same color and } 2008 \mid (x+y+z)\}
\]

\[
S_2 = \{(x,y,z) \in M^3 : x,y,z \text{ have three pairwise different colors and } 2008 \mid (x+y+z)\}
\]
Prove that $2|S_1| > |S_2|$. 