Vietnamese IMO Team Selection Test 2006

First Day

1. In an acute triangle ABC with orthocenter H, the external bisector of $\angle BHC$ meets the sides AB and AC at points D and E respectively. The internal bisector of $\angle BAC$ meets the circumcircle of $\triangle ADE$ again at K. Prove that HK bisects the side BC.

2. Find all pairs of integers (n,k) with $n \geq 0$, $k > 1$ for which the number
$$A = 17^{2006n} + 4 \cdot 17^{2n} + 7 \cdot 19^{5n}$$
is the product of k consecutive positive integers.

3. In space are given 2006 distinct points, no four of which lie on a plane. Every two points are joined by a segment. A natural number m is called *good* if each segment can be labelled with a positive integer not exceeding m so that, in each triangle, two of the labels are equal and less than the third. Find the smallest good number m.

Second Day

1. Prove that for all real numbers $x, y, z \in [1,2]$ the following inequality holds:
$$(x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) \geq 6\left(\frac{x}{y+z}+\frac{y}{z+x}+\frac{z}{x+y}\right).$$
When does equality occur?

2. A scalene triangle ABC is inscribed in a circle with center O and radius R. A variable line l is perpendicular to OA and intersects the rays AB, AC at points M, N respectively. The lines BN and CM intersect at point K, and the lines AK and BC intersect at P.

   (a) Show that the circumcircle of each triangle MNP passes through a fixed point.

   (b) Let H be the orthocenter of triangle AMN. Denote $BC = a$, and denote by d the distance from A to the line HK. Prove that $d \leq \sqrt{4R^2 - a^2}$ and show that the equality holds if and only if l, AO and BC pass through a single point.

3. The sequence $(a_n)_{n=0}^{\infty}$ is defined by $a_0 = 1$ and
$$a_{n+1} = \frac{1}{2}\left(a_n + \frac{1}{3a_n}\right).$$
Show that, for each n, $A_n = \dfrac{3}{3a_n^2 - 1}$ is a perfect square having at least n distinct prime divisors.

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