# Vietnamese IMO Team Selection Test 2004

*First Day*

1. For a set $S=\{a_1,a_2,\dots,a_{2004}\}$ with $a_1<\dots<a_{2004}$, let $f(a_i)$ denote the number of elements of $S$ that are coprime with $a_i$. Suppose that $f(a_1)=\dots=f(a_{2004})<2003$. Find the smallest positive integer $k$ such that for every set $S$ with the described properties, every $k$-element subset of $S$ contains two elements that are not coprime.

2. Find all real numbers $\alpha$ for which there is a unique function $f:\mathbb{R}\to\mathbb{R}$ satisfying

$$f(x^2+y+f(y))=f(x)^2+\alpha y \quad \text{for all } x,y\in\mathbb{R}.$$

3. Two circles $\Gamma_1$ and *Gamma*$_2$ in the plane intersect each other at $A$ and $B$. The tangents to $\Gamma_1$ at $A$ and $B$ meet at $K$. Let $M\neq A,B$ be an arbitrary point on $\Gamma_1$. The line $MK$ meets $\Gamma_1$ again at $C$, and the lines $MA$ and $CA$ meet $\Gamma_2$ again at $P$ and $Q$, respectively.

   (a) Prove that the midpoint of $PQ$ lies on the line $MC$.

   (b) Show that all lines $PQ$ pass through a single point as $M$ varies.

*Second Day*

4. The sequence $(x_n)$ is defined by $x_1=603$, $x_2=102$ and

$$x_{n+2}=x_n+x_{n+1}+2\sqrt{x_nx_{n+1}-2} \quad \text{for } n\in\mathbb{N}.$$

   (a) Prove that $x_n$ is a positive integer for all $n$.

   (b) Prove that there are infinitely many terms $x_n$ whose decimal representations end with 2003.

   (c) Prove that there is no $x_n$ whose decimal representation ends with 2004.

5. Let $A_1,B_1,C_1,D_1,E_1,F_1$ be the midpoints of the sides $AB$, $BC$, $CD$, $DE$, $EF$, $FA$ respectively of a hexagon $ABCDEF$. Let $p$ be the perimeter of hexagon $ABCDEF$ and $p_1$ be that of $A_1B_1C_1D_1E_1F_1$. Suppose that the hexagon $A_1B_1C_1D_1E_1F_1$ has equal angles. Prove that $p\geq\frac{2}{\sqrt{3}}p_1$. When does equality hold?

6. A finite set $S$ of positive integers is such that its greatest and smallest element are coprime. For each $n\in\mathbb{N}$, let $S_n$ denote the set of natural numbers which can be represented as a sum of at most $n$ elements of $S$ (not necessarily diferent). Prove that if $a$ is the greatest element of $S$, then there is an integer $b$ such that $|S_n|=an+b$ for all sufficiently large $n$.



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