Vietnamese IMO Team Selection Test 2003

First Day

1. Four positive integers \( m, n, p, q \) with \( p < m \) and \( q < n \) are given. Consider the points \( A(0,0), B(p,0), C(m,q) \) and \( D(m,n) \) in the coordinate plane. Consider the paths \( f \) from \( A \) to \( D \) and \( g \) from \( B \) to \( C \), consisting of unit steps going to the right or upwards. Let \( S \) be the number of couples \((f,g)\) such that \( f \) and \( g \) have no common points. Prove that

\[
S = \left( \frac{m+n}{n} \right) \left( \frac{m+q-p}{q} \right) - \left( \frac{m+q}{q} \right) \left( \frac{m+n-p}{n} \right).
\]

2. Let \( A_0, B_0, C_0 \) respectively be the midpoints of the altitudes \( AH, BK \) and \( CL \) of a non-equilateral triangle \( ABC \). Let \( O \) be the circumcenter and \( I \) the incenter of the triangle. The incircle of \( \triangle ABC \) touches \( BC \) at \( D \), \( CA \) at \( E \), and \( AB \) at \( F \). Show that the four lines \( A_0D, B_0E, C_0F \) and \( OI \) are concurrent.

3. A function \( f \) satisfies \( f(0,0) = 5^{2003} \), \( f(0,n) = 0 \) for all \( n \in \mathbb{N} \), and

\[
f(m,n) = f(m-1,n) - 2 \left[ \frac{f(m-1,n-1)}{2} \right] + \left[ \frac{f(m-1,n+1)}{2} \right]
\]

for all integers \( m > 0 \) and \( n \). Show that there exists a positive integer \( M \) such that \( f(M,n) = 1 \) for all integers \( n \) with \( |n| \leq \frac{(5^{2003} - 1)}{2} \) and \( f(M,n) = 0 \) for all other integers \( n \).

Second Day

4. Let \( M, N, P \) be the midpoints of the sides \( BC, CA, AB \) respectively of a triangle \( ABC \), and let \( M_1, N_1, P_1 \) be the points on the perimeter of the triangle such that each of the lines \( MM_1, NN_1, PP_1 \) bisects the perimeter.
   
   (a) Prove that the lines \( MM_1, NN_1, PP_1 \) have a common point \( K \).
   
   (b) Show that at least one of the ratios \( KA/BC, KB/CA, KC/AB \) is not less than \( 1/\sqrt{3} \).

5. Let \( A \) be the set of all permutations of the numbers \( 1, 2, \ldots, 2003 \) that fix no proper subset of \( \{1, \ldots, 2003\} \). For each permutation \( a = (a_1,a_2,\ldots,a_{2003}) \in A \), denote

\[
d(a) = \sum_{k=1}^{2003} (a_k - k)^2.
\]

   (a) Find the minimum value \( d_0 \) of \( d(a) \).
   
   (b) Find all permutations \( a \in A \) for which \( d(a) = d_0 \).

6. Prove that for any positive integer \( n \), the number \( 2^n + 1 \) has no prime divisors of the form \( 8k - 1 \), where \( k \in \mathbb{N} \).