Vietnamese IMO Team Selection Test 2003

First Day

1. Four positive integers m, n, p, q with $p < m$ and $q < n$ are given. Consider the points $A(0,0)$, $B(p,0)$, $C(m,q)$ and $D(m,n)$ in the coordinate plane. Consider the paths f from A to D and g from B to C, consisting of unit steps going to the right or upwards. Let S be the number of couples (f,g) such that f and g have no common points. Prove that
$$S = \binom{m+n}{n}\binom{m+q-p}{q} - \binom{m+q}{q}\binom{m+n-p}{n}.$$

2. Let A_0, B_0, C_0 respectively be the midpoints of the altitudes AH, BK and CL of a non-equilateral triangle ABC. Let O be the circumcenter and I the incenter of the triangle. The incircle of $\triangle ABC$ touches BC at D, CA at E, and AB at F. Show that the four lines A_0D, B_0E, C_0F and OI are concurrent.

3. A function f satisfies $f(0,0) = 5^{2003}$, $f(0,n) = 0$ for all $n \in \mathbb{N}$, and
$$f(m,n) = f(m-1,n) - 2\left[\tfrac{f(m-1,n)}{2}\right] + \left[\tfrac{f(m-1,n-1)}{2}\right] + \left[\tfrac{f(m-1,n+1)}{2}\right]$$
for all integers $m > 0$ and n. Show that there exists a positive integer M such that $f(M,n) = 1$ for all integers n with $|n| \le \dfrac{(5^{2003}-1)}{2}$ and $f(M,n) = 0$ for all other integers n.

Second Day

4. Let M, N, P be the midpoints of the sides BC, CA, AB respectively of a triangle ABC, and let M_1, N_1, P_1 be the points on the perimeter of the triangle such that each of the lines MM_1, NN_1, PP_1 bisects the perimeter.
   (a) Prove that the lines MM_1, NN_1, PP_1 have a common point K.
   (b) Show that at least one of the ratios KA/BC, KB/CA, KC/AB is not less than $1/\sqrt{3}$.

5. Let A be the set of all permutations of the numbers $1, 2, \dots, 2003$ that fix no proper subset of $\{1, \dots, 2003\}$. For each permutation $a = (a_1, a_2, \dots, a_{2003}) \in A$, denote
$$d(a) = \sum_{k=1}^{2003} (a_k - k)^2.$$
   (a) Find the minimum value d_0 of $d(a)$.
   (b) Find all permutations $a \in A$ for which $d(a) = d_0$.

6. Prove that for any positive integer n, the number $2^n + 1$ has no prime divisors of the form $8k - 1$, where $k \in \mathbb{N}$.

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