Vietnamese IMO Team Selection Test 2001

First Day – Hanoi, May 8

1. The sequence of integers \((a_n)\) is defined by \(a_0 = 1\) and
\[ a_n = a_{n-1} + a_{[n/3]}, \quad \text{for every } n \in \mathbb{N}. \]

Prove that for each prime number \(p \leq 13\) there exists \(k\) such that \(a_k\) is divisible by \(p\).

2. Two circles intersect each other at points \(A\) and \(B\). Let \(l\) be a common tangent of the two circles, touching them at \(P\) and \(T\). The tangents to the circumcircle of triangle \(APT\) at \(P\) and \(T\) meet at \(S\). Let \(H\) be the reflection of point \(B\) across the line \(l\). Prove that \(A, S, H\) are collinear.

3. There are 42 members in a club. Among any 31 of them, there is a pair consisting of a man and a woman who know each other. Prove that there are at least 12 disjoint man-woman pairs who know each other.

Second Day – Hanoi, May 9

4. Let \(x, y, z\) be positive real numbers such that \(21xy + 2yz + 8zx \leq 12\). Find the minimum value of
\[ f(x, y, z) = \frac{1}{x} + \frac{2}{y} + \frac{3}{z}. \]

5. Let \(n > 1\) be an integer. Denote by \(\mathcal{A}\) the set of points \((x, y, z)\), where \(x, y, z \in \{1, 2, \ldots, n\}\). Some points in \(\mathcal{A}\) are colored in such a manner that if point \(M(x_0, y_0, z_0)\) is colored, then point \(N(x_1, y_1, z_1)\) with \(x_1 \leq x_0, y_1 \leq y_0, z_1 \leq z_0\) is not colored. Find, with proof, the maximum possible number of colored points.

6. Let \((a_n)_{n \in \mathbb{N}}\) be a sequence of positive integers satisfying the condition
\[ 0 < a_{n+1} - a_n \leq 2001 \quad \text{for all } n \in \mathbb{N}. \]

Prove that there exist infinitely many pairs of positive integers \((p, q)\) such that \(p < q\) and \(a_p\) divides \(a_q\).

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
www.imomath.com