Vietnamese IMO Team Selection Test 2000

First Day

1. Two circles $C_1$ and $C_2$ intersect at points $P$ and $Q$. Their common tangent, closer to $P$ than to $Q$, touches $C_1$ at $A$ and $C_2$ at $B$. The tangents to $C_1$ and $C_2$ at $P$ meet the other circle at points $E \neq P$ and $F \neq P$, respectively. Let $H$ and $K$ be the points on the rays $AF$ and $BE$ respectively such that $AH = AP$ and $BK = BP$. Prove that $A, H, Q, K, B$ lie on a circle.

2. Let $k$ be a given positive integer. Define $x_1 = 1$ and, for each $n > 1$, set $x_{n+1}$ to be the smallest positive integer not belonging to the set

$$\{x_i, x_i + ik \mid i = 1, \ldots, n\}.$$ 

Prove that there is a real number $a$ such that $x_n = [an]$ for all $n \in \mathbb{N}$.

3. Two players alternately replace the stars in the expression

$$\ast x^{2000} + \ast x^{1999} + \cdots + \ast x + 1$$

by real numbers. The player who makes the last move loses if the resulting polynomial has a real root $t$ with $|t| < 1$, and wins otherwise. Give a winning strategy for one of the players.

Second Day

4. Let $a, b, c$ be pairwise coprime natural numbers. A positive integer $n$ is said to be stubborn if it cannot be written in the form

$$n = bcx + cay + abz, \quad \text{for some } x, y, z \in \mathbb{N}.$$ 

Determine the number of stubborn numbers.

5. Let $a > 1$ and $r > 1$ be real numbers.

(a) Prove that if $f : \mathbb{R}^+ \to \mathbb{R}^+$ is a function satisfying the conditions

(i) $f(x)^2 \leq ax^r f(x/a)$ for all $x > 0$,

(ii) $f(x) < x^{2000}$ for all $x < 1/2^{2000}$,

then $f(x) \leq x^a a^{1-r}$ for all $x > 0$.

(b) Construct a function $f : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying condition (i) such that for all $x > 0$, $f(x) > x^a a^{1-r}$.

6. A collection of 2000 congruent circles is given on the plane such that no two circles are tangent and each circle meets at least two other circles. Let $N$ be the number of points that belong to at least two of the circles. Find the smallest possible value of $N$. 

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