37-th Vietnamese Mathematical Olympiad 1999

First Day – March 12

1. Solve the system of equations
$$\begin{aligned}(1+4^{2x-y})5^{1-2x+y} &= 1+2^{2x-y+1};\\ y^3+4x+1+\ln(y^2+2x) &= 0.\end{aligned}$$

2. Let A',B',C' be the midpoints of the arcs BC,CA,AB of the circumcircle of a triangle ABC, not containing A,B,C, respectively. The sides BC,CA and AB meet the pairs of segments $C'A',A'B'$; $A'B',B'C'$ and $B'C',C'A'$ at M,N; P,Q and R,S respectively. Prove that $MN=PQ=RS$ if and only if the triangle ABC is equilateral.

3. The sequences (x_n) and (y_n) are defined recursively as follows:
$$\begin{aligned} x_0=1,\quad x_1=4,\quad x_{n+2}=3x_{n+1}-x_n,\\ y_0=1,\quad y_1=2,\quad y_{n+2}=3y_{n+1}-y_n,\end{aligned}\quad \text{for all } n\geq 0.$$
   (a) Prove that $x_n^2-5y_n^2+4=0$ for all $n\geq 0$.
   (b) Suppose that a,b are positive integers satisfying $a^2-5b^2+4=0$. Prove that there exists $k\geq 0$ such that $x_k=a$ and $y_k=b$.

Second Day – March 13

4. Let a,b,c be positive real numbers such that $abc+a+c=b$. Find the greatest possible value of
$$P=\frac{2}{a^2+1}-\frac{2}{b^2+1}+\frac{3}{c^2+1}.$$

5. Let Ox,Oy,Oz,Ot be rays in space, not all in the same plane, such that the angles between any two of them have the same measure.
   (a) Determine this common measure.
   (b) Let a ray Or, different from these four rays, form angles $\alpha,\beta,\gamma,\delta$ with Ox,Oy,Oz,Ot, respectively. Denote
$$p=\cos\alpha+\cos\beta+\cos\gamma+\cos\delta,\ \ q=\cos^2\alpha+\cos^2\beta+\cos^2\gamma+\cos^2\delta.$$
   Prove that p and q remain constant when Or varies.

6. Let $\mathbb{T}$ denote the set of nonnegative integers not greater than 1999. Find all functions $f:\mathbb{N}_0\to\mathbb{T}$ which satisfy
$$\begin{aligned} f(t)&=t &&\text{for all } t\in\mathbb{T},\\ f(m+n)&=f(f(m)+f(n)) &&\text{for all } m,n\in\mathbb{N}.\end{aligned}$$

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