36-th Vietnamese Mathematical Olympiad 1998

First Day – March 13

1. Let $a \geq 1$ be a real number. A sequence (x_n) is defined by

$$x_1 = a; \qquad x_{n+1} = 1 + \ln\left(\frac{x_n^2}{1+\ln x_n}\right) \quad \text{for all } n \geq 1.$$

   Prove that the sequence x_n has a finite limit and find this limit.

2. Let $ABCD$ be a tetrahedron and AA_1, BB_1, CC_1, DD_1 be diameters of its circumsphere. Let A_0, B_0, C_0, D_0 be the centroids of the triangles BCD, CDA, DAB, ABC, respectively. Prove that:

   (a) the lines A_0A_1, B_0B_1, C_0C_1 and D_0D_1 have a common point F;

   (b) the line passing through F and the midpoint of an edge is perpendicular to the opposite edge.

3. The sequence (a_n) is defined recursively by $a_0 = 20$, $a_1 = 100$ and $a_{n+2} = 4a_{n+1} + 5a_n + 20$ for $n \geq 0$. Find the smallest positive integer h for which $a_{n+h} - a_h$ is divisible by 1998 for all n.

Second Day – March 14

4. Prove that there does not exist a sequence $(x_n)_{n=1}^{\infty}$ of real numbers satisfying the following two conditions:

   (i) $|x_n| \leq 0.666$ for all $n \in \mathbb{N}$;

   (ii) $|x_n - x_m| \geq \dfrac{1}{n(n+1)} + \dfrac{1}{n(n+1)}$ whenever $m \neq n$.

5. If x, y are real numbers, determine the minimum value of the expression

$$F(x,y) = \sqrt{(x+1)^2 + (y-1)^2} + \sqrt{(x-1)^2 + (y+1)^2} + \sqrt{(x+2)^2 + (y+2)^2}.$$

6. Find all positive integers n for which there exists a polynomial $P(x)$ with real coefficients satisfying

$$P(x^{1998} - x^{-1998}) = x^n - x^{-n} \qquad \text{for all real } x \neq 0.$$

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