35-th Vietnamese Mathematical Olympiad 1997

First Day

1. Let $\mathscr{S}$ be a circle with center O and radius R, and P be a fixed point in the plane with $OP = d < R$. Let $ABCD$ be a variable quadrilateral whose diagonals AC and BD are perpendicular and meet at P. Find the maximum and minimum value of the perimeter of the quadrilateral $ABCD$ in terms of R and d.

2. Let $n > 1$ be an integer not divisible by 1997. Define

$$a_i = i + \frac{in}{1997}, \qquad i = 1, 2, \dots, 1996;$$
$$b_j = j + \frac{1997j}{n}, \quad j = 1, 2, \dots, n-1.$$

   Let $(c_k)_{k=1}^{n+1995}$ be the increasing sequence formed of the a_i's and b'_js. Prove that $c_{k+1} - c_k < 2$ for all k.

3. How many functions $f : \mathbb{N} \to \mathbb{N}$ are there which satisfy

$$f(1) = 1 \quad \text{and} \quad f(n)f(n+2) = f(n+1)^2 + 1997 \quad \text{for all } n?$$

Second Day

4. (a) Find a polynomial P with rational coefficients of the minimum degree such that $P\left(\sqrt[3]{3} + \sqrt[3]{9}\right) = 3 + \sqrt[3]{3}$.

   (b) Does there exist a polynomial Q with integer coefficients such that $Q\left(\sqrt[3]{3} + \sqrt[3]{9}\right) = 3 + \sqrt[3]{3}$?

5. Prove that for each positive integer n there is a positive integer m such that $19^m - 97$ is divisible by 2^n.

6. Let be given 75 points inside a unit cube, no three of which are collinear. Prove that there exists a triangle with vertices in these points and area not exceeding $7/72$.

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