34-th Vietnamese Mathematical Olympiad 1996

First Day

1. Solve the system of equations:

$$\sqrt{3x}\left(1+\frac{1}{x+y}\right)=2, \qquad \sqrt{7y}\left(1-\frac{1}{x+y}\right)=4\sqrt{2}.$$

2. Let be given a trihedral angle $Sxyz$. A plane π, not passing through S, cuts Sx, Sy, Sz at A, B, C respectively. On the plane π, outside $\triangle ABC$, are constructed triangles DAB, EBC, FCA congruent to SAB, SBC, SCA respectively. A sphere τ inside $Sxyz$, but outside $SABC$, touches the planes containing the faces of $SABC$. Prove that the point of tangency between τ and π is the circumcenter of triangle DEF.

3. Let k, n be integers with $1 \le k \le n$. Find the number of ordered k-tuples $(a_1, a_2, \dots, a_k)$ of distinct elements of the set $\{1, 2, \dots, n\}$ such that:

   (i) there are indices s, t such that $s < t$ and $a_s > a_t$;

   (ii) there exists s such that $a_s - s$ is odd.

Second Day

4. Find all functions $f : \mathbb{N} \to \mathbb{N}$ satisfying

$$f(n) + f(n+1) = f(n+2)f(n+3) - 1996 \quad \text{for all } n \in \mathbb{N}.$$

5. If ABC is a triangle with $BC = 1$ and $\angle A = \alpha$, find the shortest distance betwen its incenter and its centroid. If $f(\alpha)$ denotes this shortest distance, find the greatest value of $f(\alpha)$ when $\pi/3 < \alpha < \pi$.

6. Suppose that a, b, c, d are nonnegative real numbers satisfying

$$2(ab+bc+cd+da+ac+bd)+abc+bcd+cda+dab=16.$$

Prove that $3(a+b+c+d) \ge 2(ab+bc+cd+da+ac+bd)$.

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