31-st Vietnamese Mathematical Olympiad 1993

First Day

1. Find the maximum and minimum values of the function
$$f(x) = x\left(1993 + \sqrt{1995 - x^2}\right)$$
on its range.

2. Let $ABCD$ be a given quadrilateral whose no two sides are parallel. A parallelogram $MNPQ$ varies so that M,N,P,Q are interior points of sides AB,BC,CD,DA respectively. Find the locus of the centroid of the parallelogram.

3. Construct a function $f : \mathbb{N} \to \mathbb{N}$ such that
$$f(f(n)) = 1993n^{1945} \quad \text{for all } n \in \mathbb{N}.$$

Second Day

4. Find all tetrahedra $ABCD$ inscribed in a given sphere for which
$$AB^2 + AC^2 + AD^2 - BC^2 - CD^2 - BD^2$$
attains its minimum.

5. Each vertex of a convex polygon $A_1A_2\dots A_{1993}$ is assigned by + or $-$. In each step we reassigne all the vertices simultaneously, putting at A_i + if the signs at A_i and A_{i+1} are the same (where $A_{1994} = A_1$), and $-$ otherwise. Prove that after several steps the position after the first step will repeat.

6. Two sequences (a_n), (b_n) are defined by $a_0 = 2$, $b_0 = 1$ and
$$a_{n+1} = \frac{2a_nb_n}{a_n + b_n}, \qquad b_{n+1} = \sqrt{a_{n+1}b_n} \qquad \text{for all } n \in \mathbb{N}.$$
Prove that these two sequences have the same limit and evaluate this limit.

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