29-th Vietnamese Mathematical Olympiad 1991

First Day

1. Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying
   \[
   \frac{f(xy) + f(xz)}{2} - f(x)f(yz) \geq \frac{1}{4}
   \]
   for all $x, y, z \in \mathbb{R}$.

2. Let $k > 1$ be an odd integer. For every positive integer $n$, let $f(n)$ be the greatest positive integer for which $2^{f(n)}$ divides $k^n - 1$. Find $f(n)$ in terms of $k$ and $n$.

3. Three mutually perpendicular rays $Ox, Oy, Oz$ and three points $A, B, C$ on $Ox, Oy, Oz$, respectively. A variable sphere $\varepsilon$ through $A, B, C$ meets $Ox, Oy, Oz$ again at $A', B', C'$, respectively. Let $M$ and $M'$ be the centroids of triangles $ABC$ and $A'B'C'$. Find the locus of the midpoint of $MM'$.

Second Day

4. 1991 students sit around a circle and play the following game. Starting from some student $A$ and counting clockwise, each student on turn says a number. The numbers are $1, 2, 3, 1, 2, 3, \ldots$. A student who says 2 or 3 must leave the circle. The game is over when there is only one student left. What position was the remaining student sitting at the beginning of the game?

5. Let $G$ be the centroid and $R$ the circumradius of a triangle $ABC$. The extensions of $GA, GB, GC$ meet the circumcircle again at $D, E, F$. Prove that
   \[
   \frac{3}{R} \leq \frac{1}{GD} + \frac{1}{GE} + \frac{1}{GF} \leq \sqrt{3} \leq \left( \frac{1}{AB} + \frac{1}{BC} + \frac{1}{CA} \right).
   \]

6. If $x \geq y \geq z \geq 0$ are real numbers, prove that
   \[
   \frac{x^2y}{z} + \frac{y^2z}{x} + \frac{z^2x}{y} \geq x^2 + y^2 + z^2.
   \]