25-th Vietnamese Mathematical Olympiad 1987

First Day

1. Let \( u_1, u_2, \ldots, u_{1987} \) be an arithmetic progression with \( u_1 = \pi / 1987 \) and the common difference \( \pi / 3974 \). Evaluate

\[
S = \sum_{\varepsilon_i \in \{-1, 1\}} \cos(\varepsilon_1 u_1 + \varepsilon_2 u_2 + \cdots + \varepsilon_{1987} u_{1987}).
\]

2. Sequences \((x_n)\) and \((y_n)\) are constructed as follows:

\[
x_0 = 365, \quad x_{n+1} = x_n(x_{n}^{1986} + 1) + 1622, \\
y_0 = 16, \quad y_{n+1} = y_n(y_{n}^3 + 1) - 1952, \quad \text{for all } n \geq 0.
\]

Prove that \(x_n \neq y_k\) for any positive integers \(n, k\).

3. Let be given \(n \geq 2\) lines on a plane, not all concurrent and no two parallel. Prove that there is a point which belongs to exactly two of the given lines.

Second Day

4. Let \(a_1, a_2, \ldots, a_n\) be positive real numbers \((n \geq 2)\) whose sum is \(S\). Show that

\[
\sum_{i=1}^{n} \frac{a_i^{2^k}}{(S-a_i)^{2^t-1}} \geq \frac{S^{1+2^k-2^t}}{(n-1)^{2^t-1}n^{2^t-2^t}}
\]

for any nonnegative integers \(k, t\) with \(k \geq t\). When does equality occur?

5. Let \(f : [0, +\infty) \to \mathbb{R}\) be a differentiable function. Suppose that

\[
|f(x)| \leq 5 \quad \text{and} \quad f(x)f'(x) \geq \sin x \quad \text{for all } x \geq 0.
\]

Prove that there exists \(\lim_{x \to +\infty} f(x)\).

6. Prove that among any five distinct rays \(Ox, Oy, Oz, Ot, Or\) in space there exist two which form an angle less than or equal to 90\(^\circ\).