24-th Vietnamese Mathematical Olympiad 1986

First Day

1. Let $1/2 \leq a_1, a_2, \ldots, a_n \leq 5$ be given real numbers and let $x_1, x_2, \ldots, x_n$ be real numbers satisfying $4x_i^2 - 4a_ix_i + (a_i - 1)^2 \leq 0$. Prove that
\[
\sqrt{n \sum_{i=1}^{n} \frac{x_i^2}{n}} \leq \sum_{i=1}^{n} \frac{x_i}{n} + 1.
\]

2. Let $R, r$ be respectively the circumradius and inradius of a regular $1986-$gonal pyramid. Prove that
\[
\frac{R}{r} \geq 1 + \frac{1}{\cos(\pi/1986)}
\]
and find the total area of the surface of the pyramid when the equality occurs.

3. Suppose $M(y)$ is a polynomial of degree $n$ such that $M(y) = 2^y$ for $y = 1, 2, \ldots, n + 1$. Compute $M(n + 2)$.

Second Day

4. Let $ABCD$ be a square of side $a$. An equilateral triangle $AMB$ is constructed in the plane through $AB$ perpendicular to the plane of the square. A point $S$ moves on $AB$. Let $P$ be the projection of $M$ on $SC$ and $E, O$ be the midpoints of $AB$ and $CM$ respectively.

(a) Find the locus of $P$ as $S$ moves on $AB$.

(b) Find the maximum and minimum lengths of $SO$.

5. Find all $n > 1$ such that the inequality
\[
\sum_{i=1}^{n} x_i^2 \geq x_n \sum_{i=1}^{n-1} x_i
\]
holds for all real numbers $x_1, x_2, \ldots, x_n$.

6. A sequence of positive integers is constructed as follows: the first term is 1, the following two terms are 2, 4, the following three terms are 5, 7, 9, the following four terms are 10, 12, 14, 16, etc. Find the $n$-th term of the sequence.