22-nd Vietnamese Mathematical Olympiad 1984

First Day

1. (a) Find a polynomial with integer coefficients of the smallest degree having $\sqrt{2}+\sqrt[3]{3}$ as a root.
   (b) Solve $1+\sqrt{1+x^2}\left(\sqrt{(1+x)^3}-\sqrt{(1-x)^3}\right)=2\sqrt{1-x^2}$.

2. The sequence (u_n) is defined by $u_1=1$, $u_2=2$ and $u_{n+1}=3u_n-u_{n-1}$ for $n\geq 2$. Set $v_n=\sum_{k=1}^{n}\operatorname{arccot} u_k$. Compute $\lim_{n\to\infty} v_n$.

3. A square $ABCD$ of side length $2a$ is given on a plane Π. Let S be a point on the ray Ax perpendicular to Π such that $AS=2a$.
   (a) Let $M\in BC$ and $N\in CD$ be two variable points.
      i. Find the positions of M,N such that $BM+DN\geq 3/2$, planes SAM and SMN are perpendicular and $BM\cdot DN$ is minimum.
      ii. Find M and N such that $\angle MAN=45^\circ$ and the volume of $SAMN$ attains an extremum value. Find these values.
   (b) Let Q be a point such that $\angle AQB=\angle AQD=90^\circ$. The line DQ intersects the plane π through AB perpendicular to Π at Q'.
      i. Find the locus of Q'.
      ii. Let $\mathscr{K}$ be the locus of points Q and let CQ meet $\mathscr{K}$ again at R. Let DR meets π at R'. Prove that $\sin^2 Q'DB+\sin^2 R'DB$ is independent of Q.

Second Day

4. (a) Let x,y be integers, not both zero. Find the minimum possible value of $|5x^2+11xy-5y^2|$.
   (b) Find all positive real numbers t such that $\dfrac{9t}{10}=\dfrac{[t]}{t-[t]}$.

5. Given two real numbers a,b with $a\neq 0$, find all polynomials $P(x)$ which satisfy
$$xP(x-a)=(x-b)P(x).$$

6. Consider a trihedral angle $Sxyz$ with $\angle xSz=\alpha$, $\angle xSy=\beta$ and $\angle ySz=\gamma$. Let A,B,C denote the dihedral angles at edges y,z,x respectively.
   (a) Prove that $\dfrac{\sin\alpha}{\sin A}=\dfrac{\sin\beta}{\sin B}=\dfrac{\sin\gamma}{\sin C}$.
   (b) Show that $\alpha+\beta=180^\circ$ if and only if $\angle A+\angle B=180^\circ$.
   (c) Assume that $\alpha=\beta=\gamma=90^\circ$. Let $O\in Sz$ be a fixed point such that $SO=a$ and let M,N be variable points on x,y respectively. Prove that $\angle SOM+\angle SON+\angle MON$ is constant and find the locus of the incenter of $OSMN$.

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