46-th Vietnamese Mathematical Olympiad 2008

January 29

1. Determine the number of solutions of simultaneous equations
$$x^2 + y^3 = 29,$$
$$\log_3 x \cdot \log_2 y = 1.$$

2. Given a triangle with acute angle $\angle BEC$, let E be the midpoint of AB. Point M is chosen on the opposite ray of EC such that $\angle BME = \angle ECA$. Denote by θ the measure of $\angle BEC$. Express MC/AB in terms of θ.

3. Let $m = 2007^{2008}$. How many natural numbers n are there such that $n < m$ and $n(2n+1)(5n+2)$ divides m?

4. The sequence of real number (x_n) is defined by
$$x_1 = 0, \;\; x_2 = 2, \qquad x_{n+2} = 2^{-x_n} + \frac{1}{2}, \qquad \text{for all } n = 1, 2, 3, \ldots.$$
Prove that the sequence has a limit as n approaches $+\infty$. Determine the limit.

5. What is the total number of natural numbers divisible by 9 the number of digits of which does not exceed 2008 and at least two of the digits are 9's?

6. Let x, y, z be distinct non-negative real numbers. Prove that
$$\frac{1}{(x-y)^2} + \frac{1}{(y-z)^2} + \frac{1}{(z-x)^2} \geq \frac{4}{xy+yz+zx}.$$
When does equality hold?

7. Let ABC be a triangle with altitude AD, line d is perpendicular to AD, and M is a variable point on d. Let E, F be the midpoints of MB, MC. The line through E perpendicular to d intersects AB at P, the line through F perpendicular to d meets AC at Q. Prove that the line through M perpendicular to PQ has a fixed point as M varies on the line d.

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